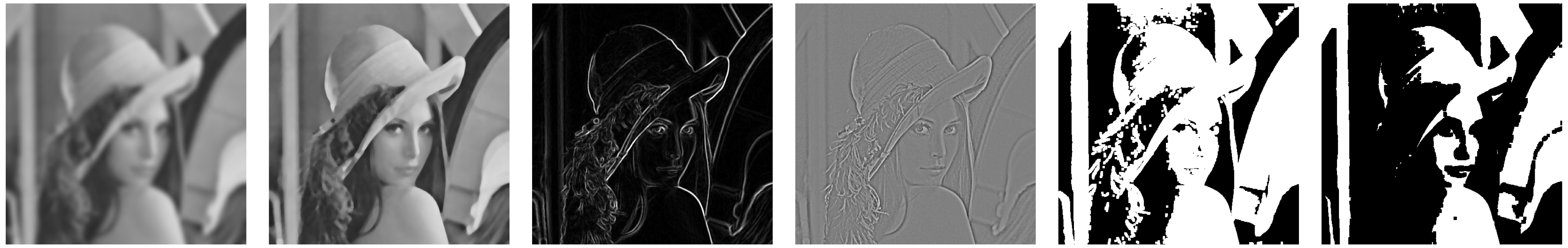
THE UNIVERSITY OF BRITISH COLUMBIA

CPSC 425: Computer Vision



Lecture 5: Image Filtering (still continued)

(unless otherwise stated slides are taken or adopted from **Bob Woodham, Jim Little** and **Fred Tung**)

Menu for Today (**September 18, 2020**)

Topics:

— **Gaussian** and **Pillbox** filters
— **Separability**
— The **Convolution Theorem**
— **Fourier** Space Representations

Redings:

— **Today's** Lecture: none
— **Next** Lecture: Forsyth & Ponce (2nd ed.) 4.4

Reminders:

— **Assignment 1:** Image Filtering and Hybrid Images due Wednsday, **Sept 30**th

— We will have our first **quiz** sometime next week (on Canvas). **Format**: Quiz is 1-2 minute per question (total time < 10 min). Can be started within 24 hour window.

Today's "**fun**" Example: Rolling Shutter

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Lecture 4: Re-cap

— The **correlation** of $F(X,Y)$ and $I(X,Y)$ is:

$$\underbrace{\boxed{I'(X,Y)}}_{\text{output}} = \sum_{j=-k}^{k} \sum_{i=-k}^{k} \underbrace{\boxed{F(i,j)}}_{\text{filter}} \underbrace{\boxed{I(X+i, Y+j)}}_{\text{image (signal)}}$$

— **Visual interpretation**: Superimpose the filter F on the image I at (X,Y), perform an element-wise multiply, and sum up the values

— **Convolution** is like **correlation** except filter “flipped”

if $F(X,Y) = F(-X,-Y)$ then correlation = convolution.

Lecture 4: Re-cap

Ways to handle **boundaries**

– **Ignore/discard**. Make the computation undefined for top/bottom k rows and left/right-most k columns

– **Pad with zeros**. Return zero whenever a value of I is required beyond the image bounds

– **Assume periodicity.** Top row wraps around to the bottom row; left wraps around to right

– **Reflection across border.** Local reflection of pixels across the top, bottom, left and right borders

Simple **examples** of filtering:

— copy, shift, smoothing, sharpening

Linear filter **properties**:

— superposition, scaling, shift invariance

Characterization Theorem: Any linear, shift-invariant operation can be expressed as a convolution

Example 5: Smoothing with a Box Filter

Original

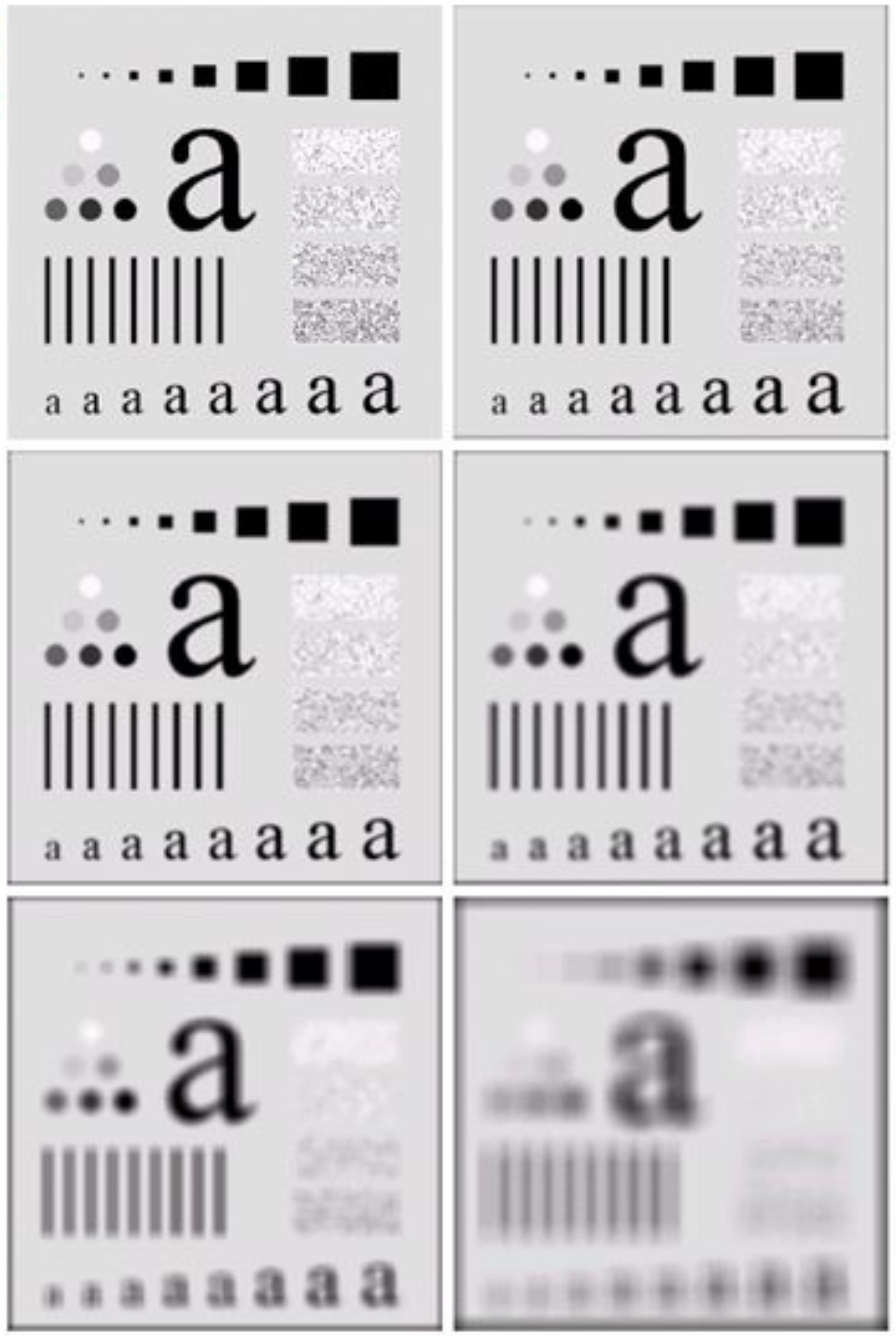
3x3

5x5

9x9

15x15

35x35



Gonzales & Woods (3rd ed.) Figure 3.3

Smoothing

Smoothing with a box **doesn’t model lens defocus** well

— Smoothing with a box filter depends on direction

— Image in which the center point is 1 and every other point is 0

Smoothing

Smoothing with a box **doesn't model lens defocus** well

— Smoothing with a box filter depends on direction

— Image in which the center point is 1 and every other point is 0

Smoothing with a (circular) **pillbox** is a better model for defocus (in geometric optics)

The **Gaussian** is a good general smoothing model

— for phenomena (that are the sum of other small effects)

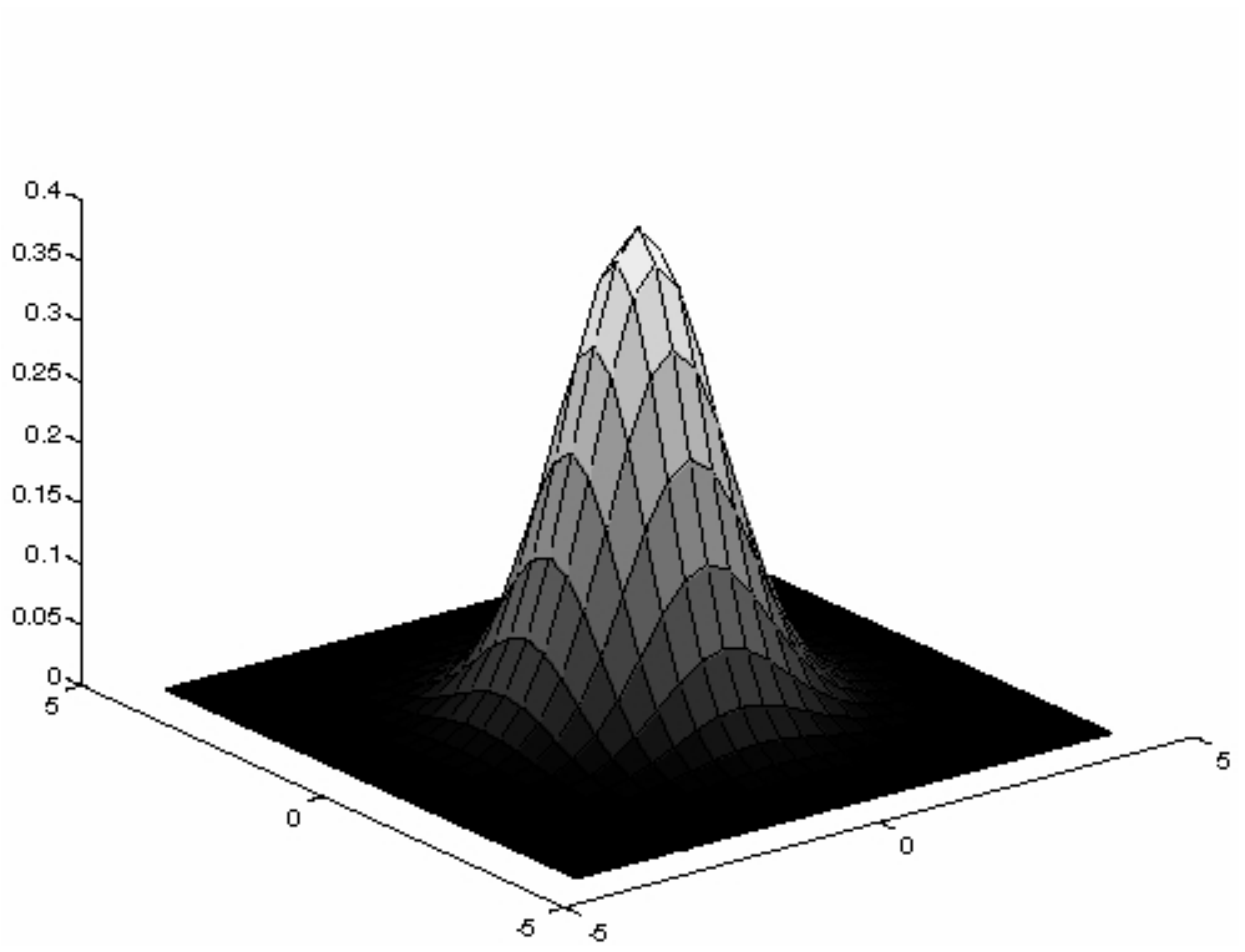
— whenever the Central Limit Theorem applies

Example 6: Smoothing with a Gaussian

Idea: Weight contributions of pixels by spatial proximity (nearness)

2D **Gaussian** (continuous case):

$$G_\sigma(x, y) = \frac{1}{2\pi\sigma^2} \exp^{-\frac{x^2+y^2}{2\sigma^2}}$$



Forsyth & Ponce (2nd ed.)
Figure 4.2

Example 6: Smoothing with a Gaussian

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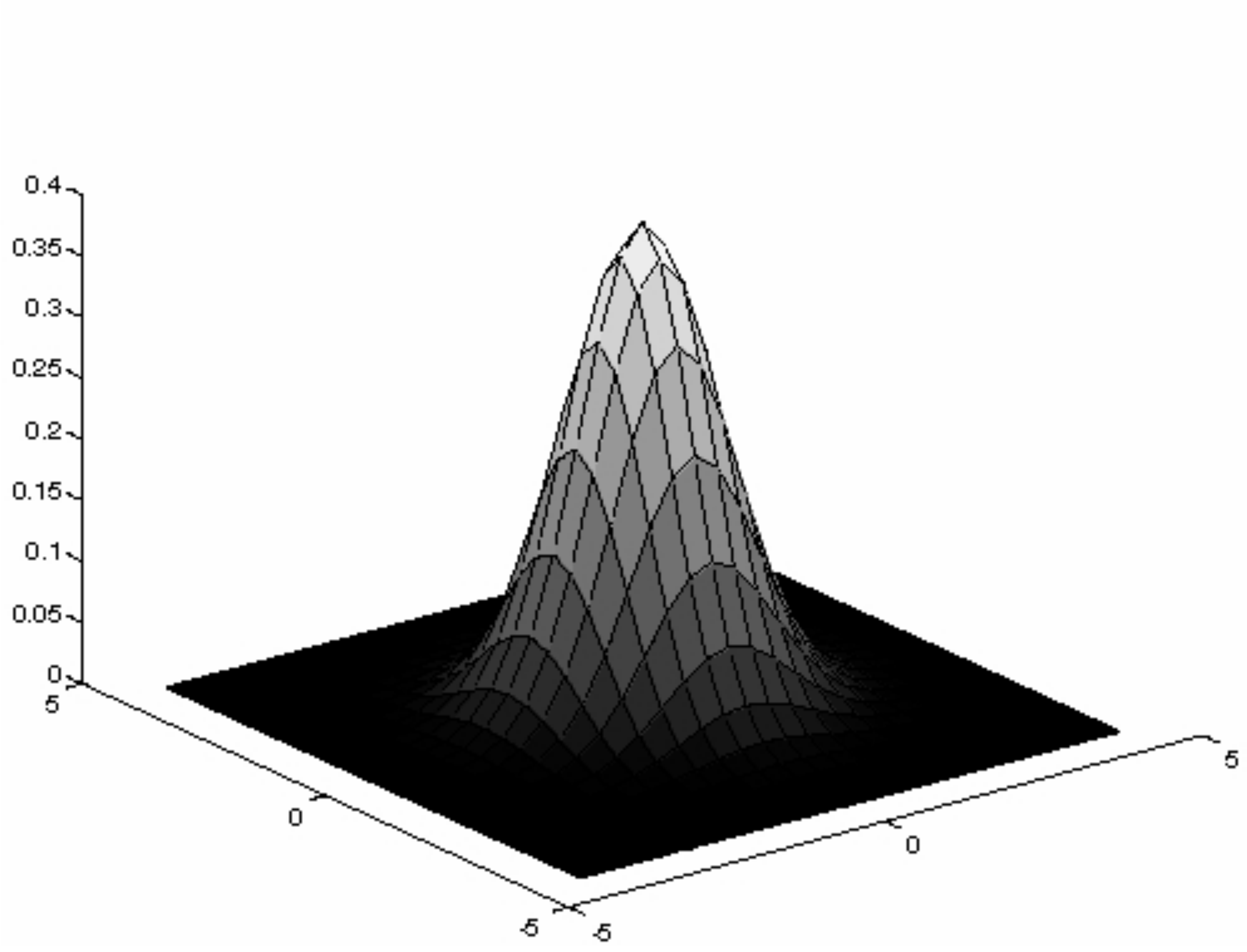
2D **Gaussian** (continuous case):

$$G_\sigma(x, y) = \frac{1}{2\pi\sigma^2} \exp^{-\frac{x^2+y^2}{2\sigma^2}}$$

Standard Deviation



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Example 6: Smoothing with a Gaussian

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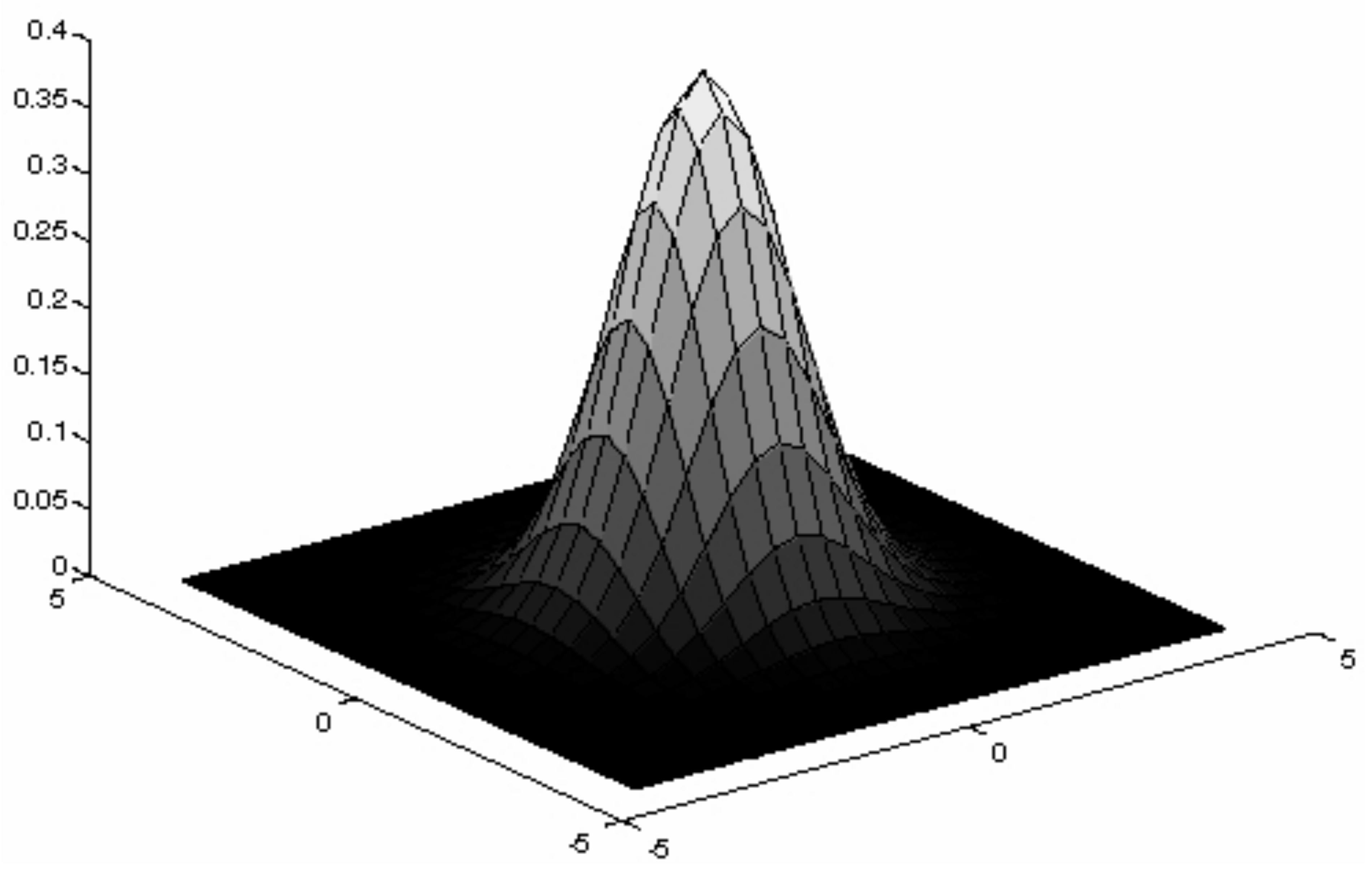

Forsyth & Ponce (2nd ed.)
Figure 4.2

1. Define a continuous **2D function**
2. **Discretize it** by evaluating this function on the discrete pixel positions to obtain a filter

Example 6: Smoothing with a Gaussian

Quantized an truncated **3x3 Gaussian** filter:

$G_\sigma(-1,1)$	$G_\sigma(0,1)$	$G_\sigma(1,1)$
$G_\sigma(-1,0)$	$G_\sigma(0,0)$	$G_\sigma(1,0)$
$G_\sigma(-1,-1)$	$G_\sigma(0,-1)$	$G_\sigma(1,-1)$

Example 6: Smoothing with a Gaussian

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With $\sigma = 1$:

0.059	0.097	0.059
0.097	0.159	0.097
0.059	0.097	0.059

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What happens if σ is larger?

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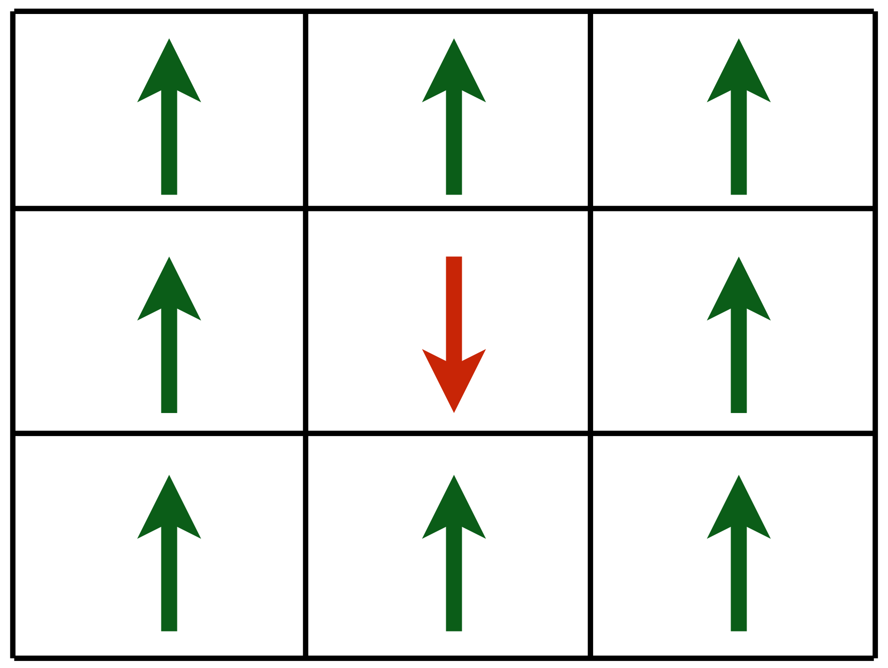

What happens if σ is larger?

— **More** blur

Example 6: Smoothing with a Gaussian

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0.097	0.159	0.097
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What happens if σ is larger?

What happens if σ is smaller?

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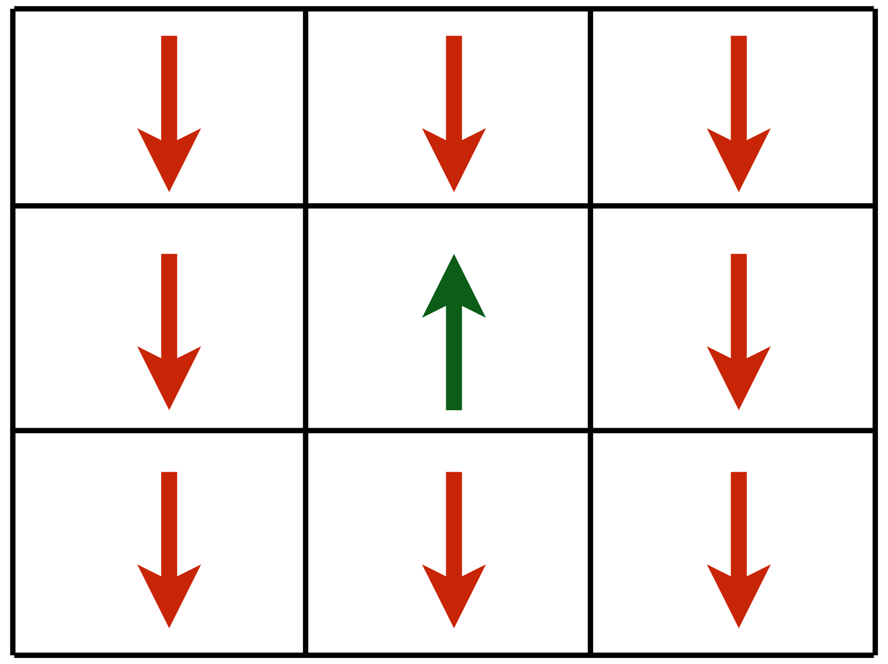
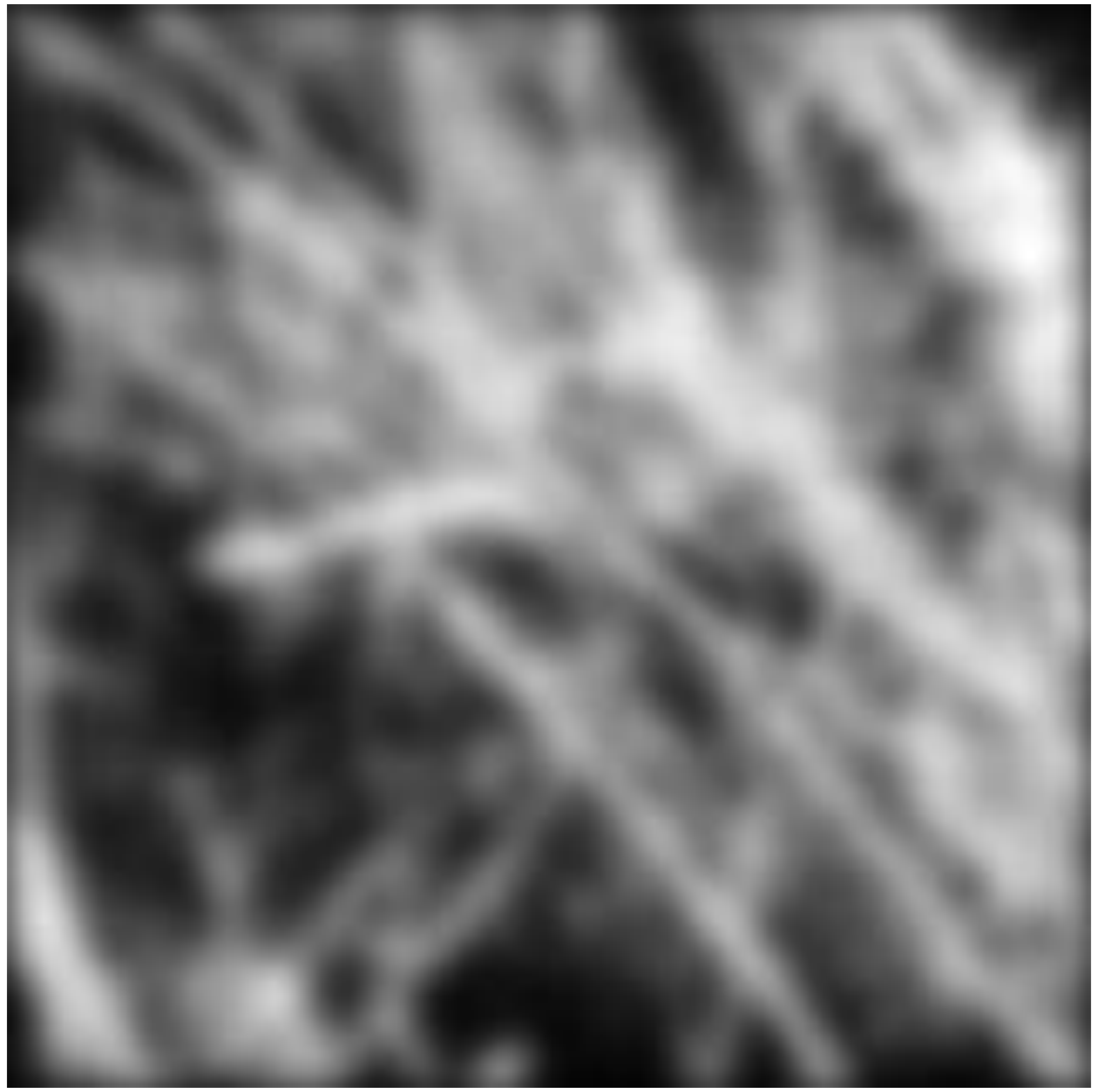


What happens if σ is larger?

What happens if σ is smaller?

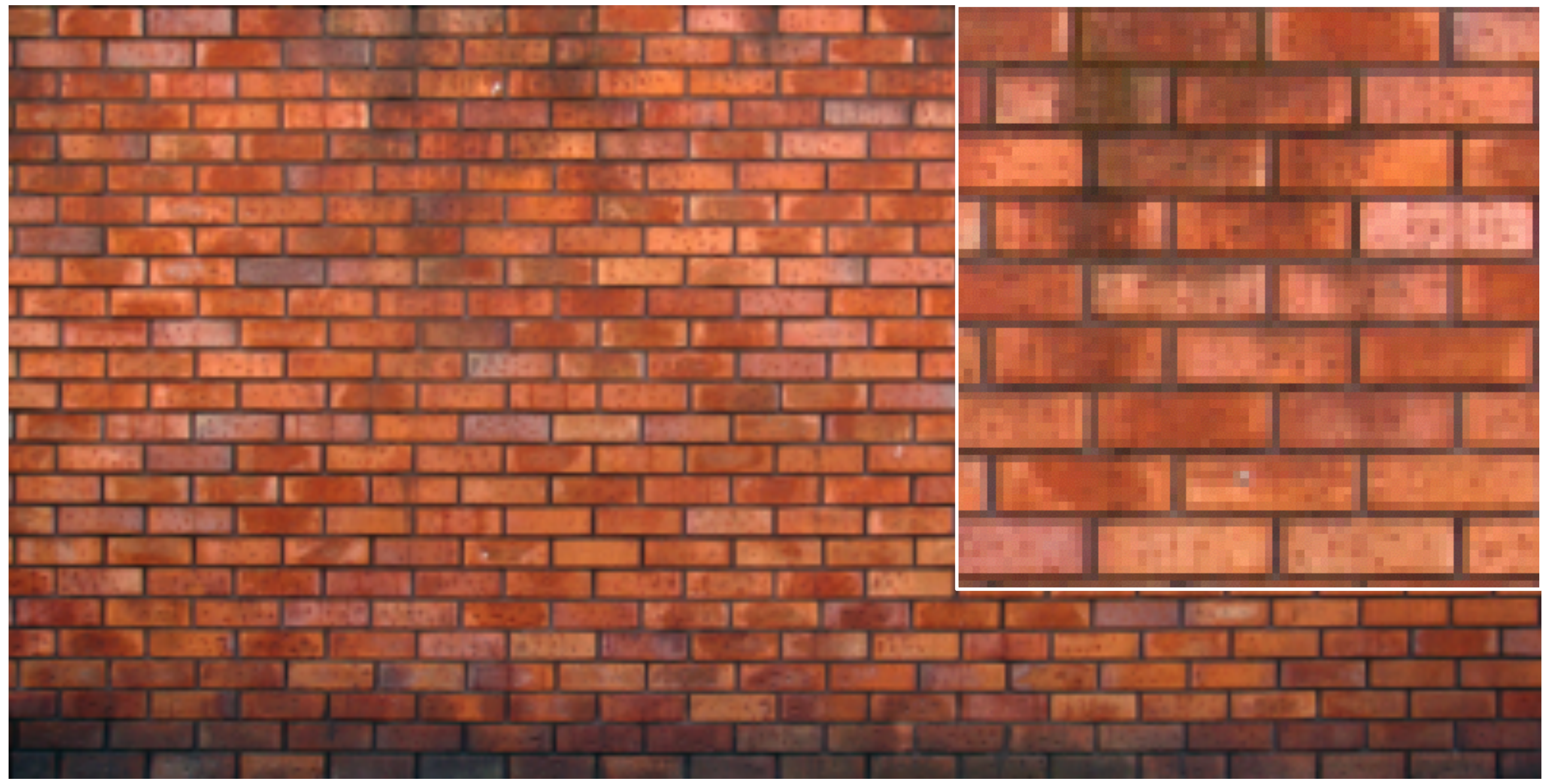
— **Less** blur

Example 6: Smoothing with a Gaussian

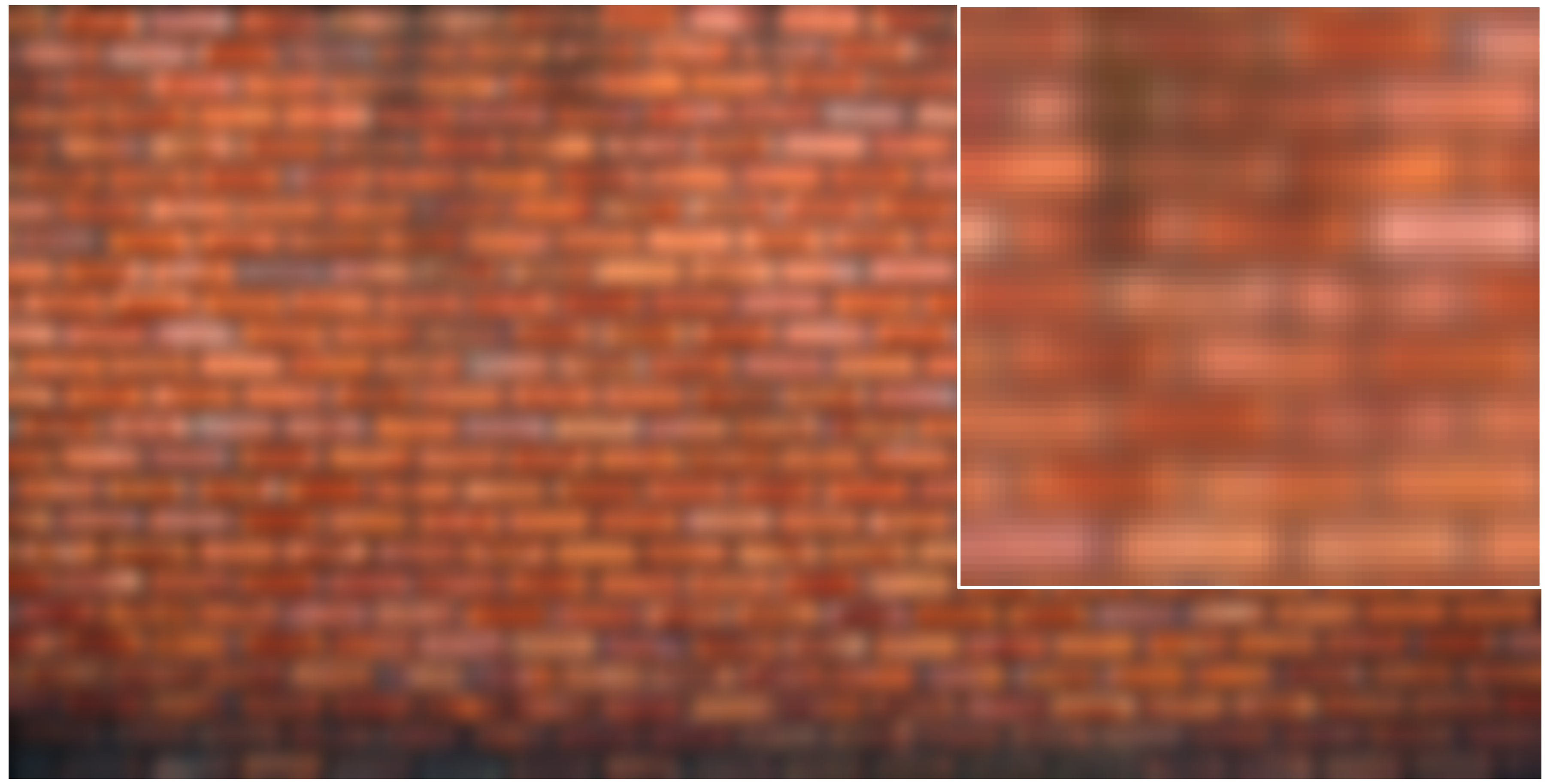


Forsyth & Ponce (2nd ed.) Figure 4.1 (left and right)

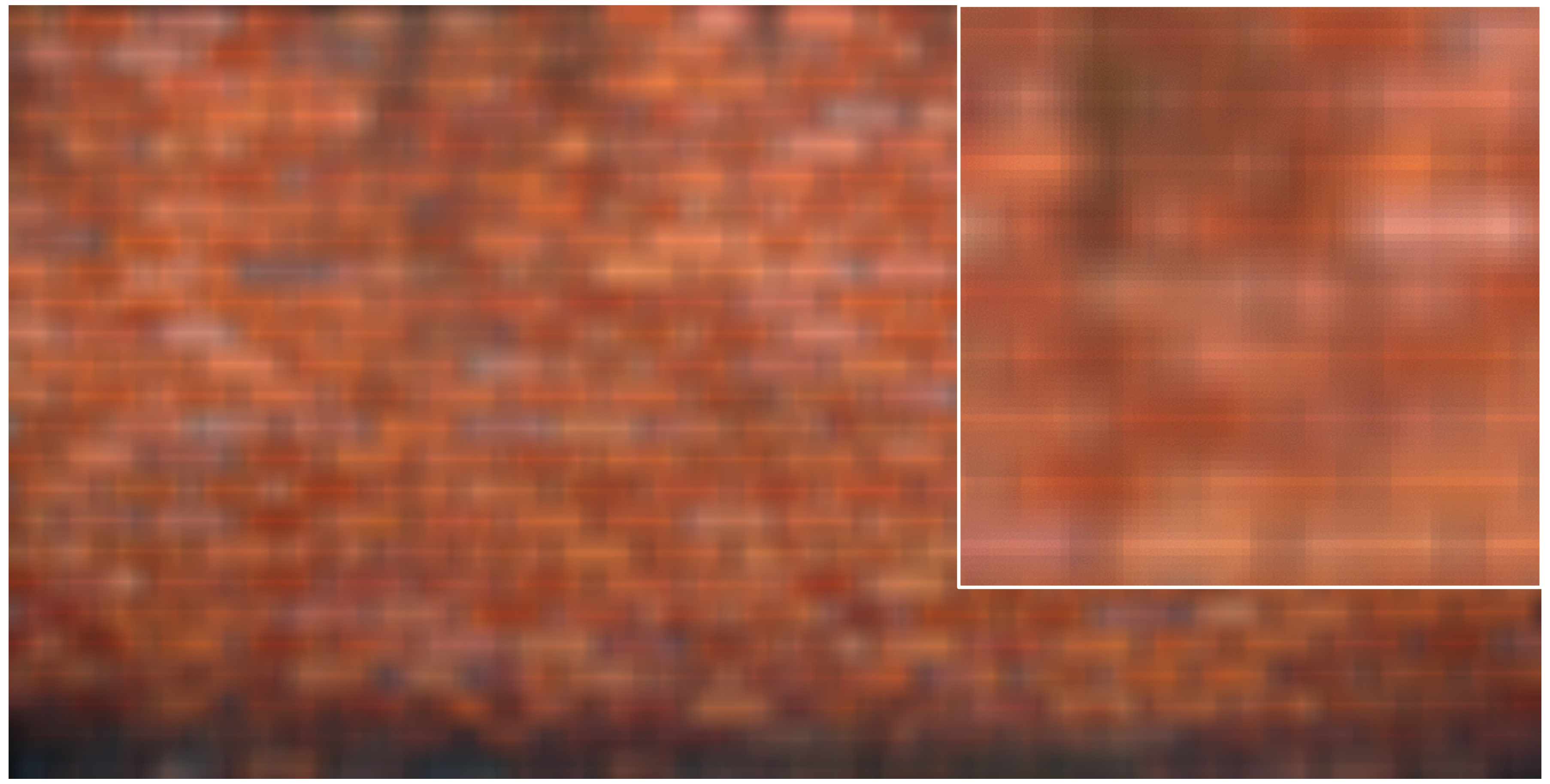
Box vs. **Gaussian** Filter



original



7x7 Gaussian



7x7 box

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

Fun: How to get shadow effect?

University of
British
Columbia

Adopted from: Ioannis (Yannis) Gkioulekas (CMU)

Fun: How to get shadow effect?

University of British Columbia

Blur with a Gaussian kernel, then compose the blurred image with the original (with some offset)

Adopted from: Ioannis (Yannis) Gkioulekas (CMU)

Example 6: Smoothing with a Gaussian

Quantized an truncated **3x3 Gaussian** filter:

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With $\sigma = 1$:

0.059	0.097	0.059
0.097	0.159	0.097
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What is the problem with this filter?

Example 6: Smoothing with a Gaussian

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With $\sigma = 1$:

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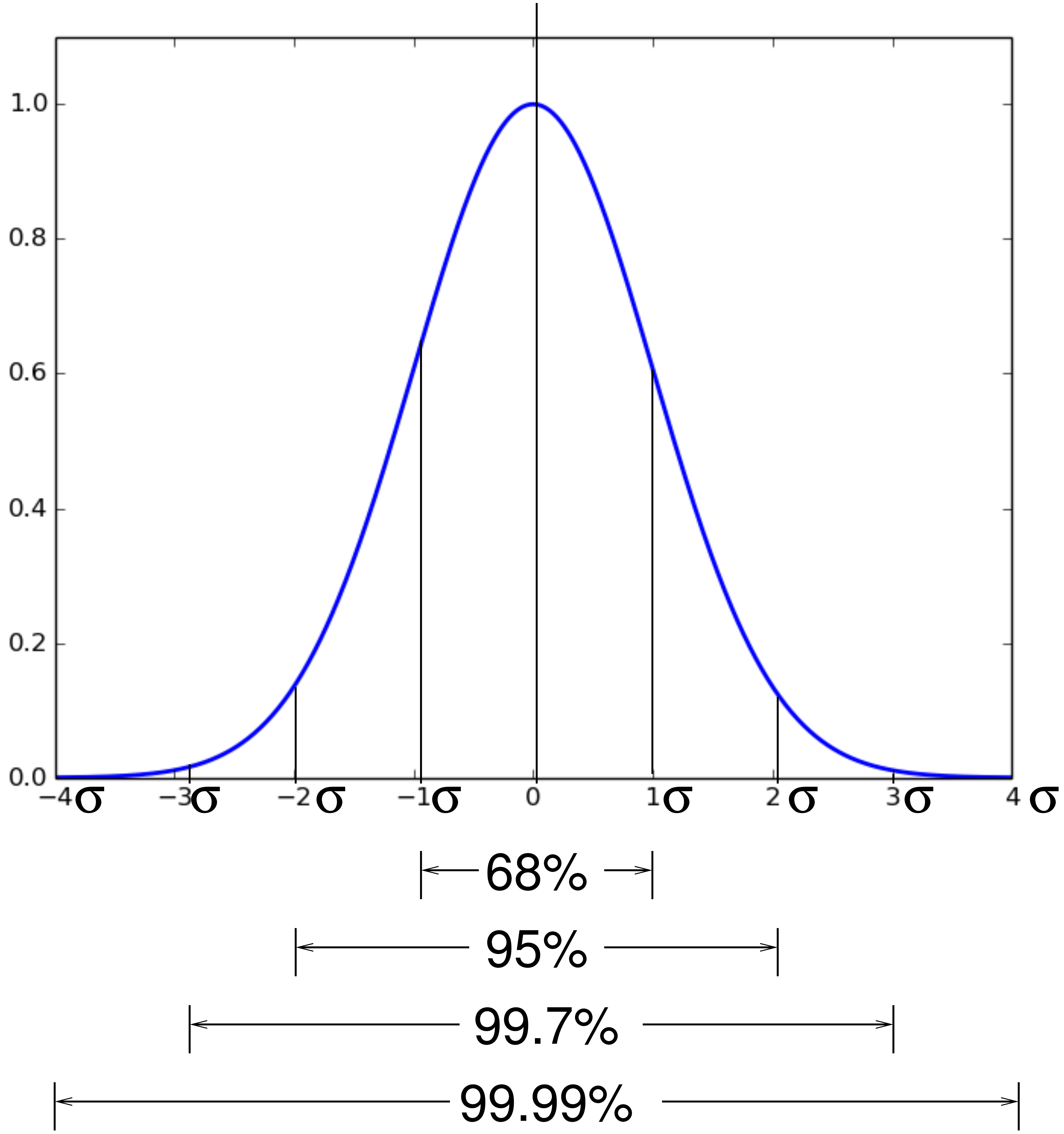
0.097	0.159	0.097
0.059	0.097	0.059

What is the problem with this filter?

does not sum to 1

truncated too much

Gaussian: Area Under the Curve



Example 6: Smoothing with a Gaussian

With $\sigma = 1$:

0.059	0.097	0.059
0.097	0.159	0.097
0.059	0.097	0.059

Better version of the Gaussian filter:

— sums to 1 (normalized)

— captures $\pm 2\sigma$

$\frac{1}{273}$

1	4	7	4	1
4	16	26	16	4
7	26	41	26	7
4	16	26	16	4
1	4	7	4	1

In general, you want the Gaussian filter to capture $\pm 3\sigma$, for $\sigma = 1$ => 7x7 filter

Lets talk about **efficiency**

Efficient Implementation: **Separability**

A 2D function of x and y is **separable** if it can be written as the product of two functions, one a function only of x and the other a function only of y

Both the **2D box filter** and the **2D Gaussian filter** are **separable**

Both can be implemented as two 1D convolutions:

— First, convolve each row with a 1D filter

— Then, convolve each column with a 1D filter

— Aside: or vice versa

The **2D Gaussian** is the only (non trivial) 2D function that is both separable and rotationally invariant.

Separability: Box Filter Example

Standard (3x3)

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$$F(X,Y) = F(X)F(Y)$$

filter

$\frac{1}{9}$

1	1	1
1	1	1
1	1	1

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	0	10	20	30	30	30	20	10	
	10	10	10	10	0	0	0	0	
	10	30	10	10	0	0	0	0	

Separability: Box Filter Example

Standard (3x3)

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$$F(X,Y) = F(X)F(Y)$$

filter

$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	0	10	20	30	30	30	20	10	
	10	10	10	10	0	0	0	0	
	10	30	10	10	0	0	0	0	

Separable

image $I(X,Y)$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$F(X)$

filter

$$\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

	0	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	0	
	0	30	60	90	90	90	60	30	
	0	30	60	90	90	90	60	30	
	0	30	30	60	60	90	60	30	
	0	30	60	90	90	90	60	30	
	0	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	0	
	30	30	30	30	0	0	0	0	
	0	0	0	0	0	0	0	0	

Separability: Box Filter Example

Standard (3x3)

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$$F(X,Y) = F(X)F(Y)$$

filter

$\frac{1}{9}$

1	1	1
1	1	1
1	1	1

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	0	10	20	30	30	30	20	10	
	10	10	10	10	0	0	0	0	
	10	30	10	10	0	0	0	0	

Separable

image $I(X,Y)$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$F(X)$

filter

$\frac{1}{3}$

1	1	1

	0	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	0	
0	30	60	90	90	90	60	30		
	0	30	60	90	90	90	60	30	
	0	30	30	60	60	90	60	30	
	0	30	60	90	90	90	60	30	
	0	0	0	0	0	0	0	0	
	0	0	0	0	0	0	0	0	
	30	30	30	30	0	0	0	0	
	0	0	0	0	0	0	0	0	

$F(Y)$

filter

$\frac{1}{3}$

1
1
1

output $I'(X,Y)$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	0	10	20	30	30	30	20	10	
	10	10	10	10	0	0	0	0	
	10	30	10	10	0	0	0	0	

Separability: How do you know if filter is separable?

If a 2D filter can be expressed as an outer product of two 1D filters

$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \odot \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Efficient Implementation: **Separability**

For example, recall the 2D **Gaussian**:

$$G_\sigma(x, y) = \frac{1}{2\pi\sigma^2} \exp^{-\frac{x^2+y^2}{2\sigma^2}}$$

The 2D Gaussian can be expressed as a product of two functions, one a function of x and another a function of y

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The 2D Gaussian can be expressed as a product of two functions, one a function of x and another a function of y

In this case the two functions are (identical) 1D Gaussians

Efficient Implementation: **Separability**

Naive implementation of 2D **Gaussian**:

At each pixel, (X,Y), there are	$m \times m$	multiplications
There are	$n \times n$	pixels in (X,Y)
Total:	$m^2 \times n^2$	multiplications

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Efficient Implementation: **Separability**

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Separable 2D **Gaussian**:

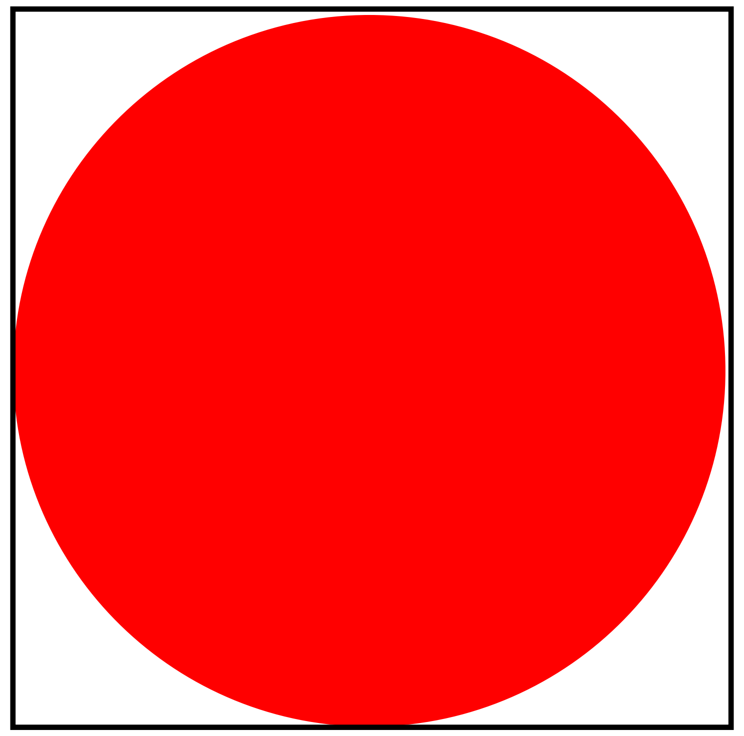
At each pixel, (X, Y), there are	$2m$	multiplications
There are	$n \times n$	pixels in (X, Y)
Total:	$2m \times n^2$	multiplications

Example 7: Smoothing with a Pillbox

Let the radius (i.e., half diameter) of the filter be r

In a contentious domain, a 2D (circular) pillbox filter, $f(x, y)$, is defined as:

$$f(x,y) = \frac{1}{\pi r^2} \begin{cases} 1 & \text{if } \; x^2 + y^2 \leq r^2 \\ 0 & \text{otherwise} \end{cases}$$



The scaling constant, $\frac{1}{\pi r^2}$, ensures that the area of the filter is one

Example 7: Smoothing with a Pillbox

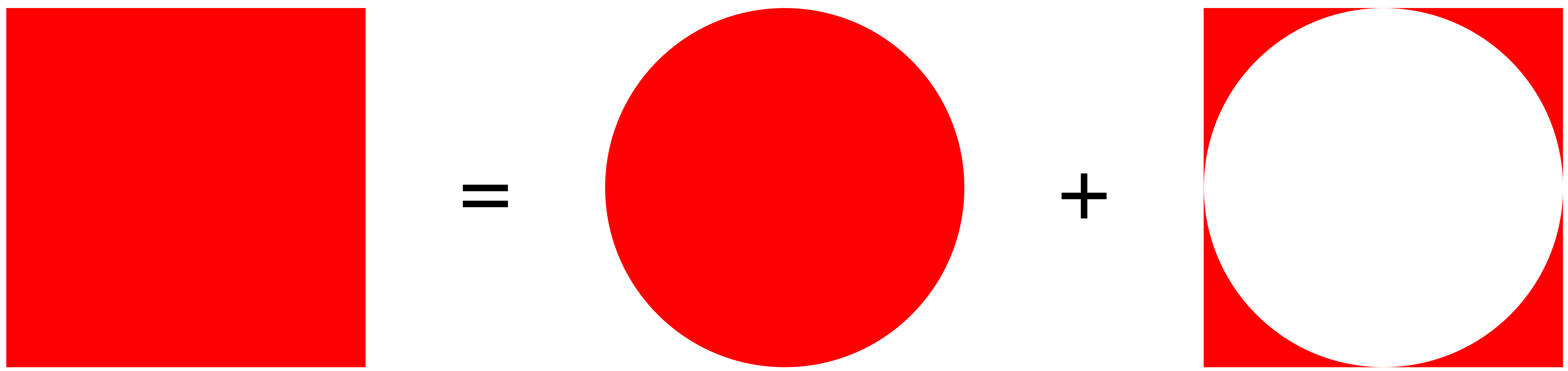
Recall that the 2D Gaussian is the only (non trivial) 2D function that is both **separable** and **rotationally invariant**.

A **2D pillbox** is rotationally invariant but not separable.

There are occasions when we want to convolve an image with a 2D pillbox. Thus, it worth exploring possibilities for **efficient implementation**.

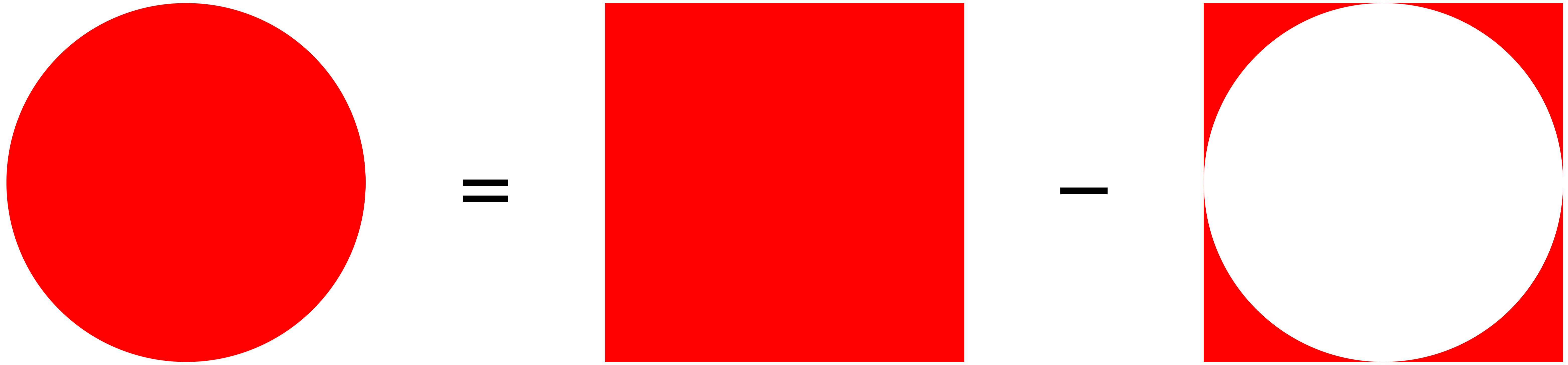
Example 7: Smoothing with a Pillbox

A 2D box filter can be expressed as the sum of a 2D pillbox and some “extra corner bits”

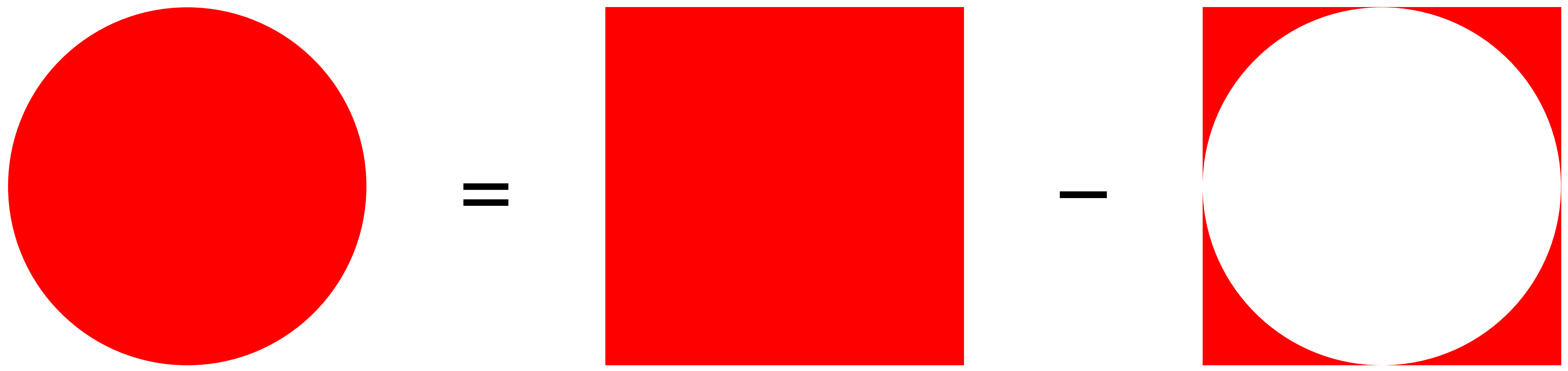


Example 7: Smoothing with a Pillbox

Therefore, a 2D pillbox filter can be expressed as the difference of a 2D box filter and those same “extra corner bits”



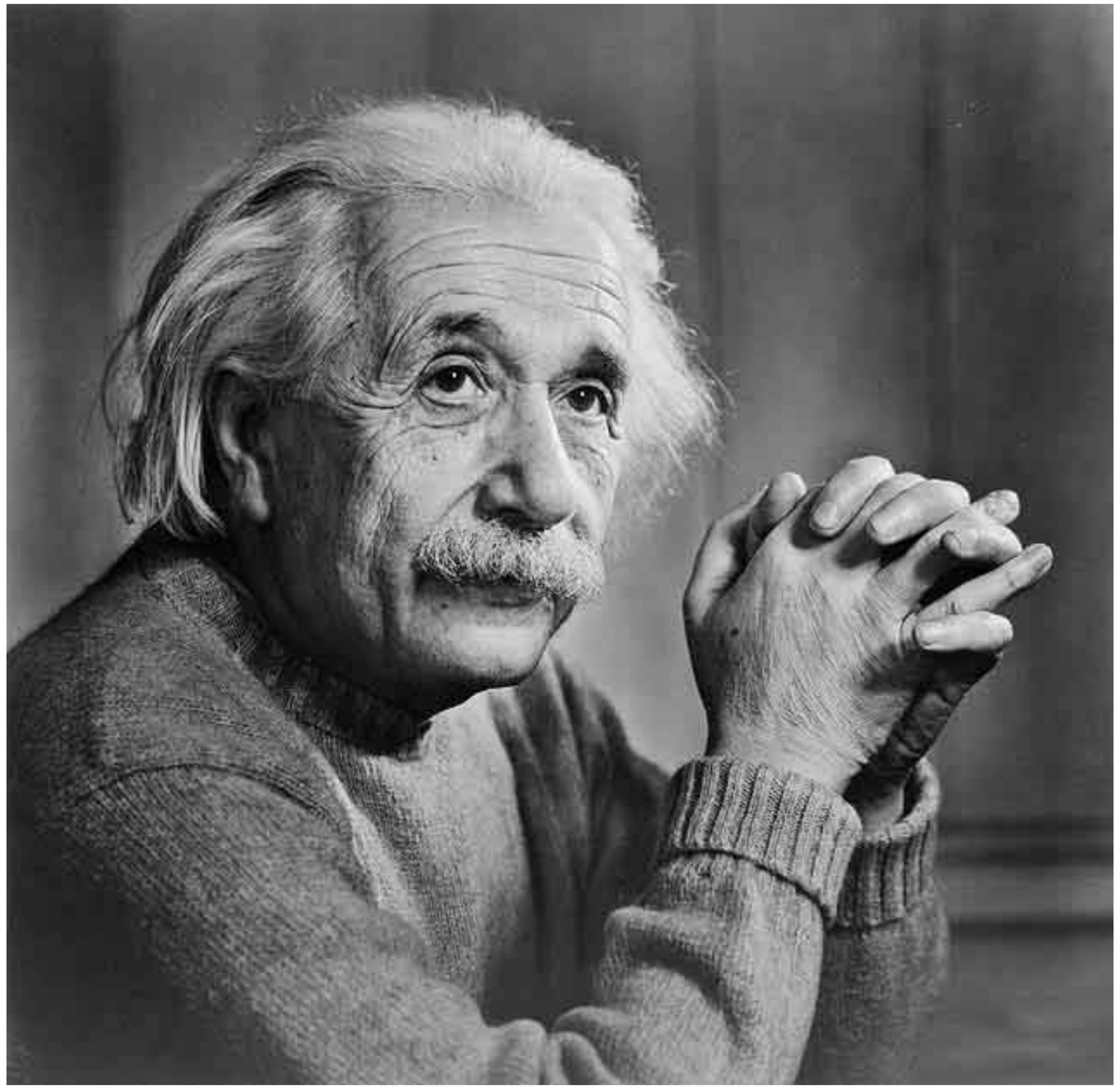
Example 7: Smoothing with a Pillbox



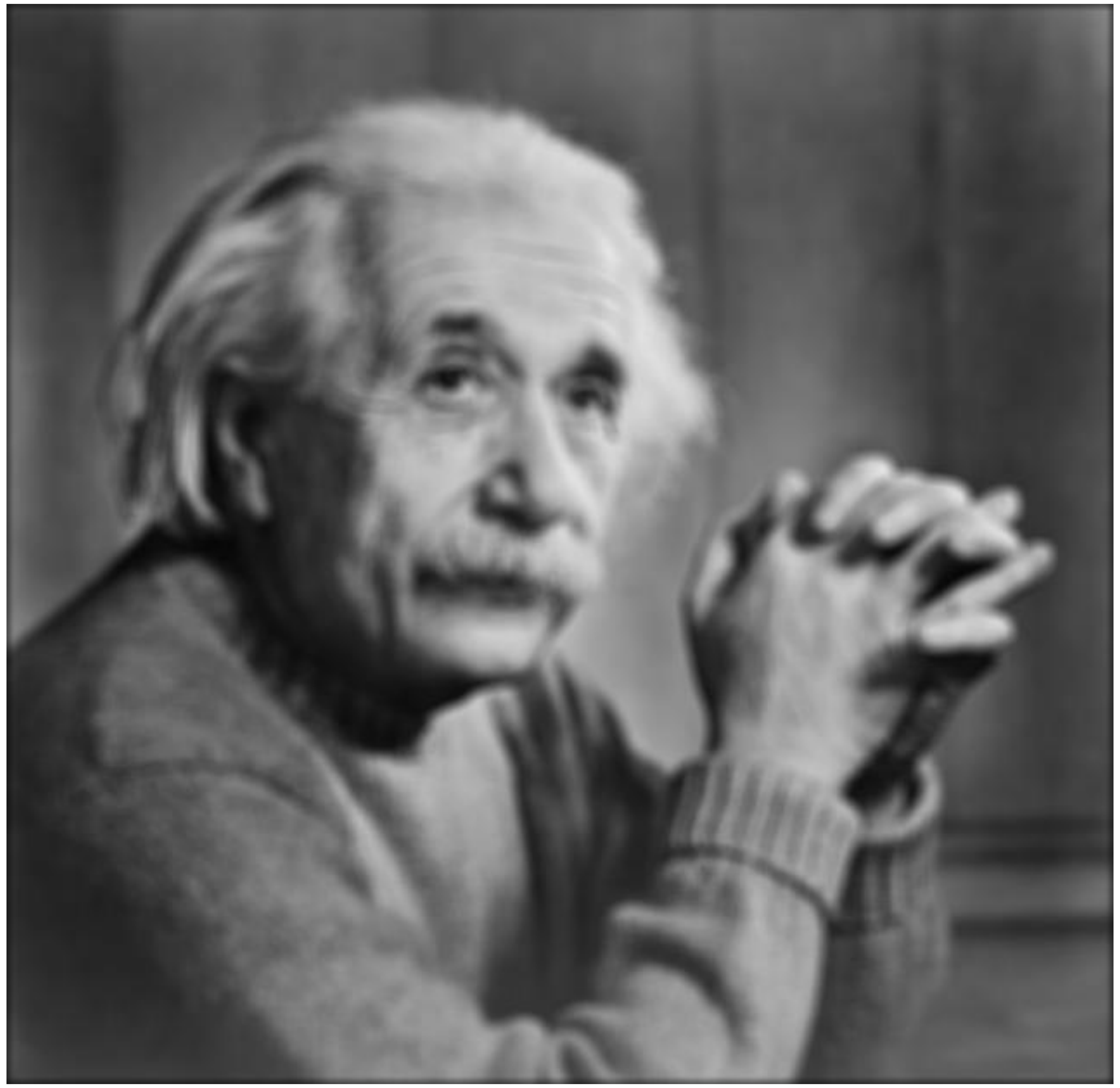
Implementing convolution with a 2D pillbox filter as the difference between convolution with a box filter and convolution with the “extra corner bits” filter allows us to take advantage of the separability of a box filter

Further, we can postpone scaling the output to a single, final step so that convolution involves filters containing all 0’s and 1’s
— This means the required convolutions can be implemented without any multiplication at all

Example 7: Smoothing with a Pillbox



Original



11 x 11 Pillbox

Speeding Up **Convolution** (The Convolution Theorem)

Let z be the product of two numbers, x and y, that is,

$$z = xy$$

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Therefore.

$$z = \exp^{\ln z} = \exp^{(\ln x + \ln y)}$$

Interpretation: At the expense of two $\ln()$ and one $\exp()$ computations, multiplication is reduced to admission

Speeding Up **Rotation**

Another analogy: **2D rotation of a point by an angle** α about the origin

The standard approach, in Euclidean coordinates, involves a matrix multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

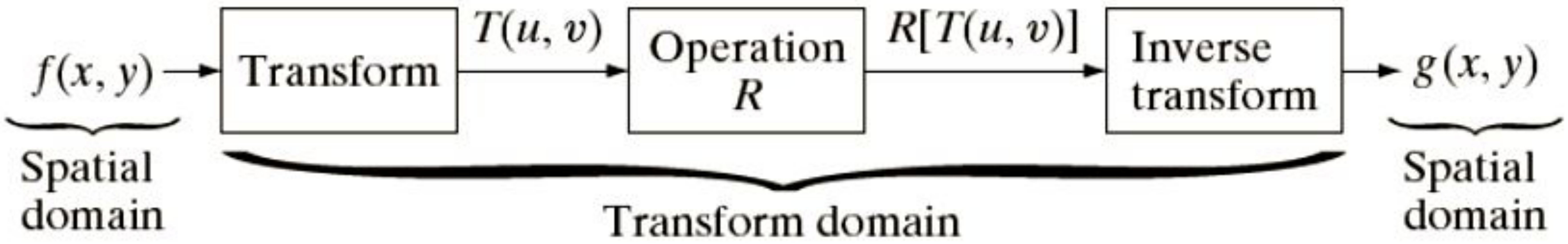
Suppose we transform to polar coordinates

$$(x, y) \rightarrow (\rho, \theta) \rightarrow (\rho, \theta + \alpha) \rightarrow (x', y')$$

Rotation becomes addition, at expense of one polar coordinate transform and one inverse polar coordinate transform

Speeding Up **Convolution** (The Convolution Theorem)

Similarly, some image processing operations become cheaper in a transform domain



Gonzales & Woods (3rd ed.) Figure 2.39

Speeding Up **Convolution** (The Convolution Theorem)

Convolution **Theorem**:

Let $$i'(x,y) = f(x,y) \otimes i(x,y)$$

then $$\mathcal{I}'(w_x,w_y) = \mathcal{F}(w_x,w_y)\,\mathcal{I}(w_x,w_y)$$

where $\mathcal{I}'(w_x,w_y)$, $\mathcal{F}(w_x,w_y)$, and $\mathcal{I}(w_x,w_y)$ are Fourier transforms of $i'(x,y)$, $f(x,y)$ and $i(x,y)$

At the expense of two **Fourier** transforms and one inverse Fourier transform, convolution can be reduced to (complex) multiplication

Lets take a **detour** …

What follows is for fun

(you will **NOT** be tested on this)

Fourier Transform (you will **NOT** be tested on this)

Basic building block:

$$A \sin(\omega x + \phi)$$

Fourier's claim: Add enough of these to get <u>any</u> periodic signal you want!

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

Fourier Transform (you will **NOT** be tested on this)

Basic building block:

$$A \sin(\omega x + \phi)$$

amplitude

sinusoid

angular frequency

variable

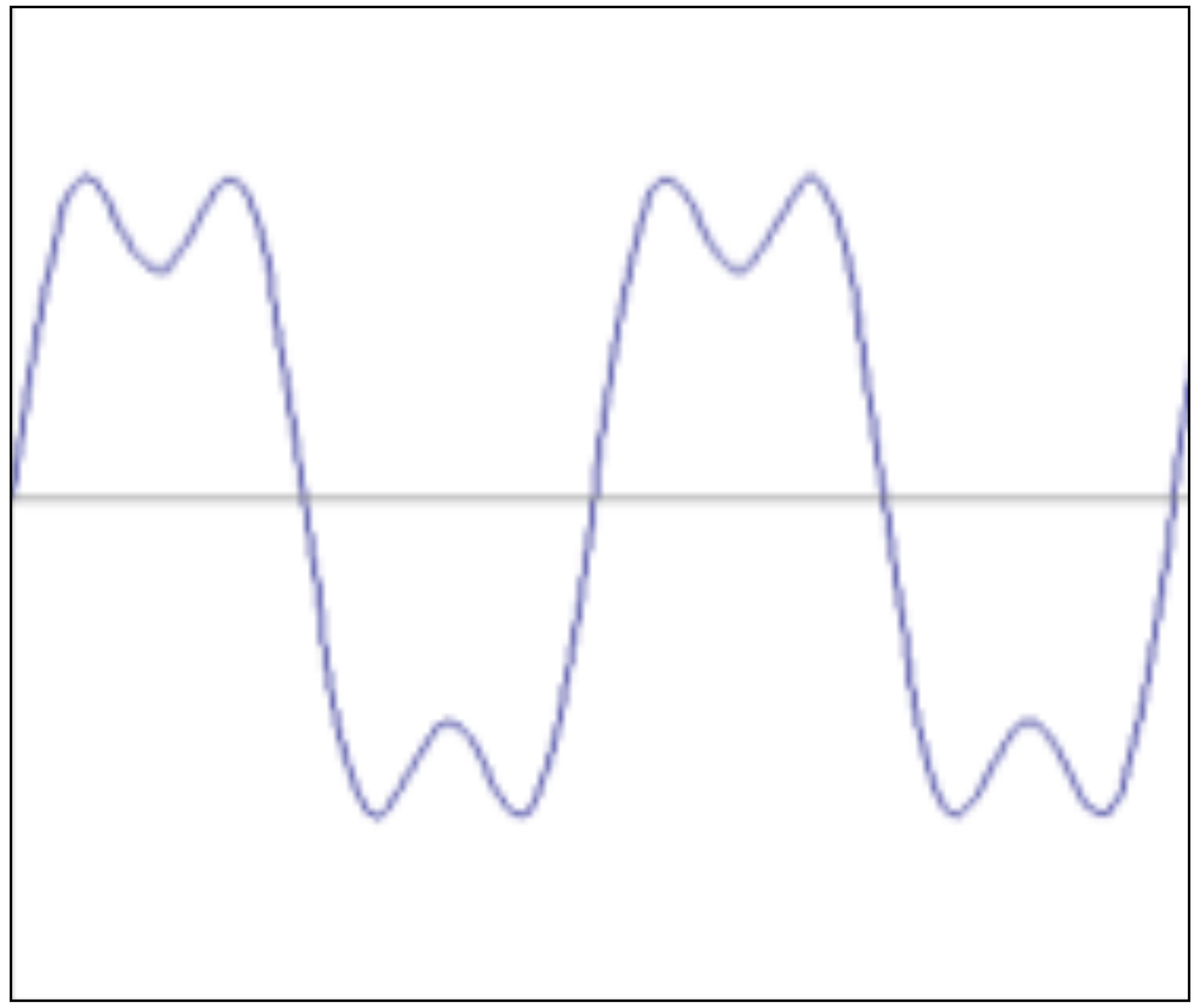
phase

Fourier's claim: Add enough of these to get any periodic signal you want!

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

Fourier Transform (you will **NOT** be tested on this)

How would you generate this function?



= ? + ?

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

Fourier Transform (you will **NOT** be tested on this)

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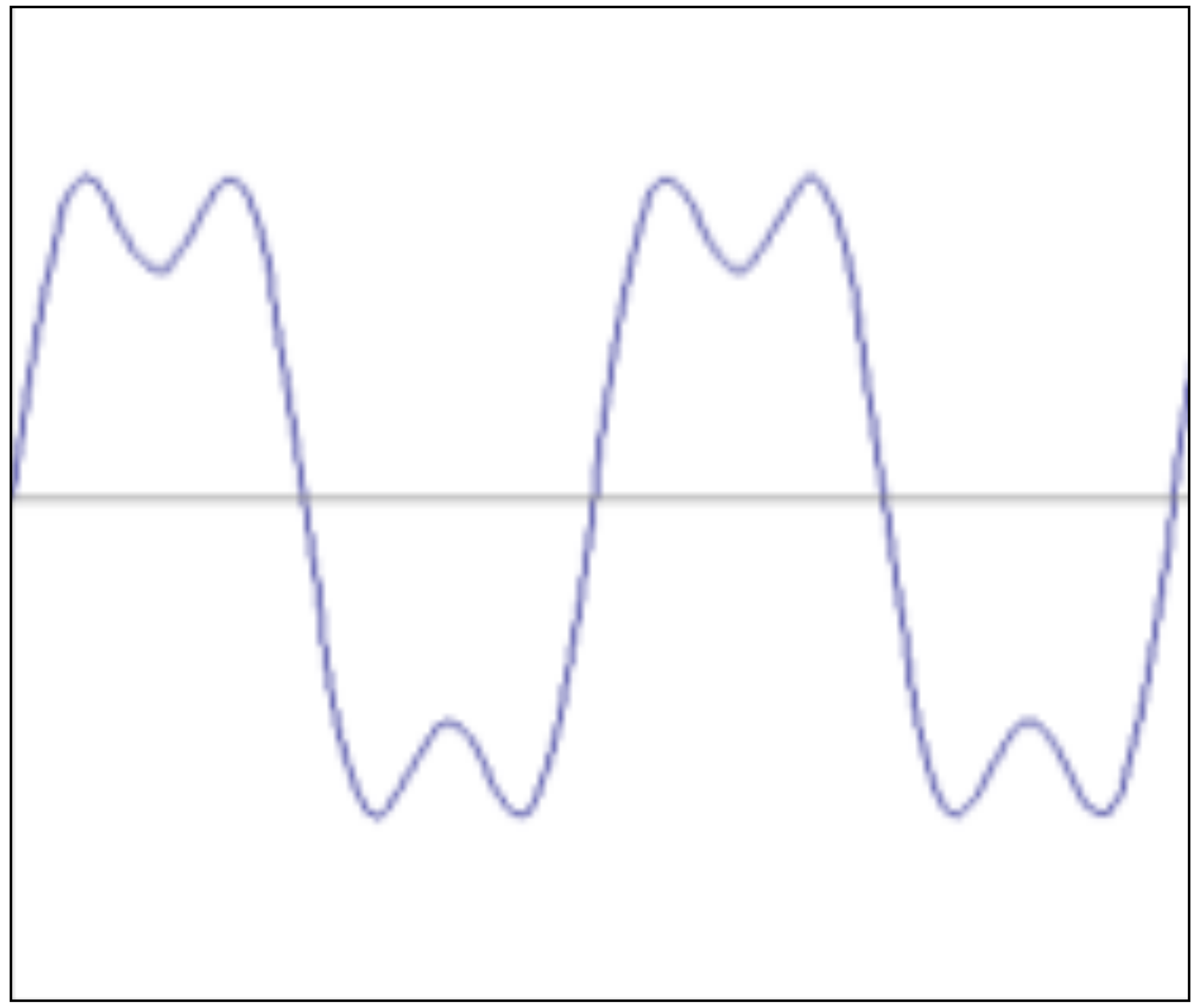
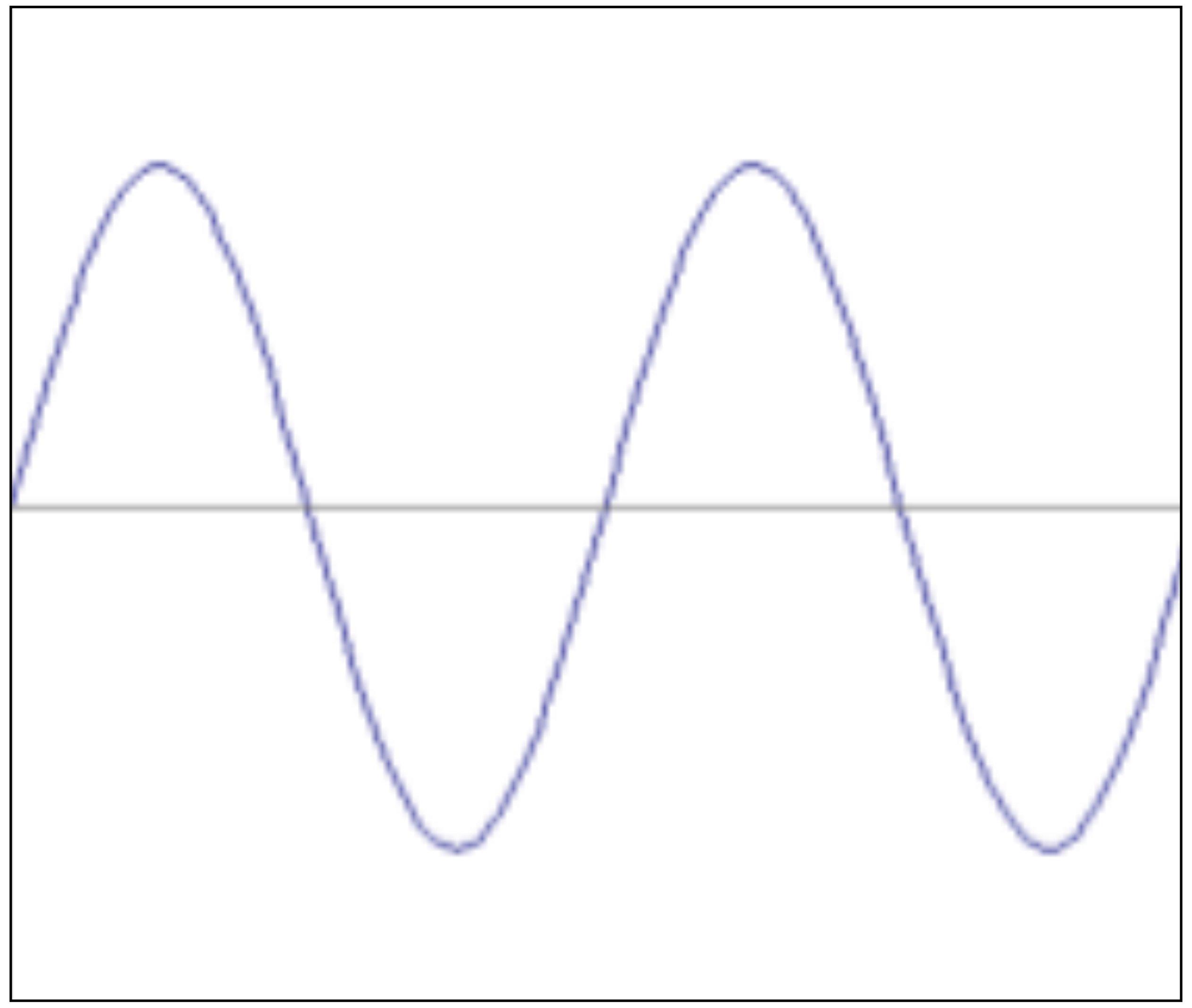


=



+ ?

$\sin(2\pi x)$

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

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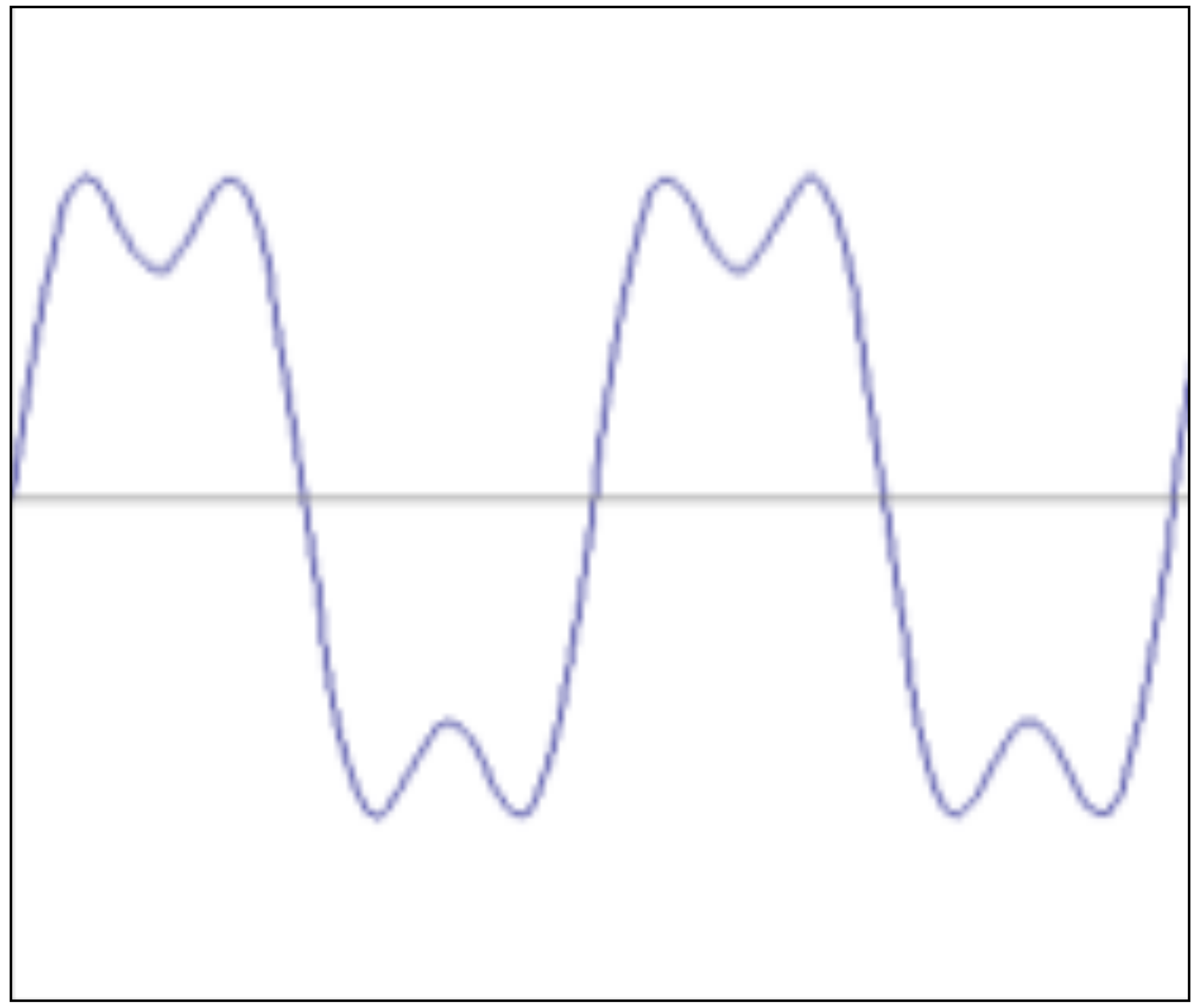
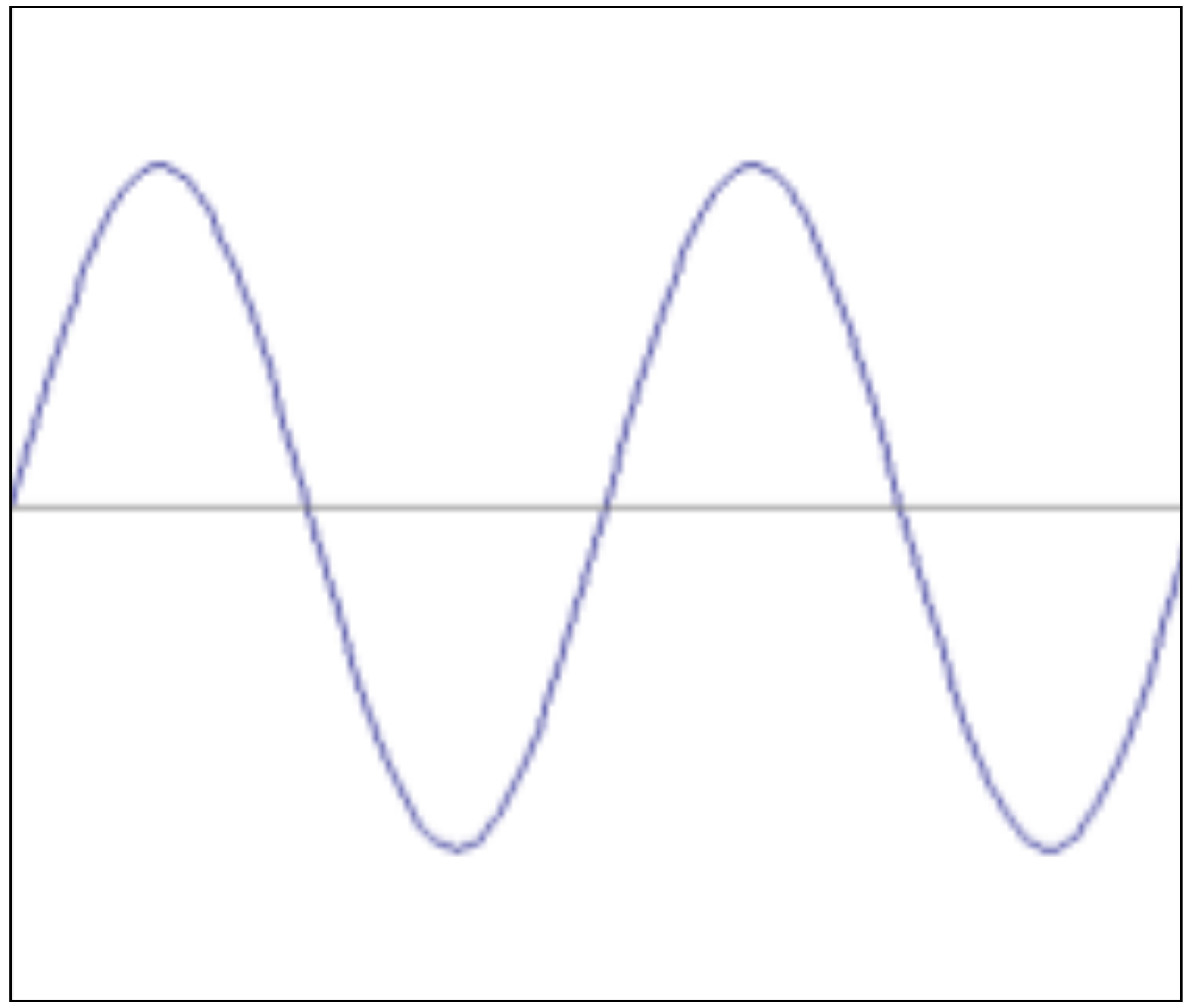
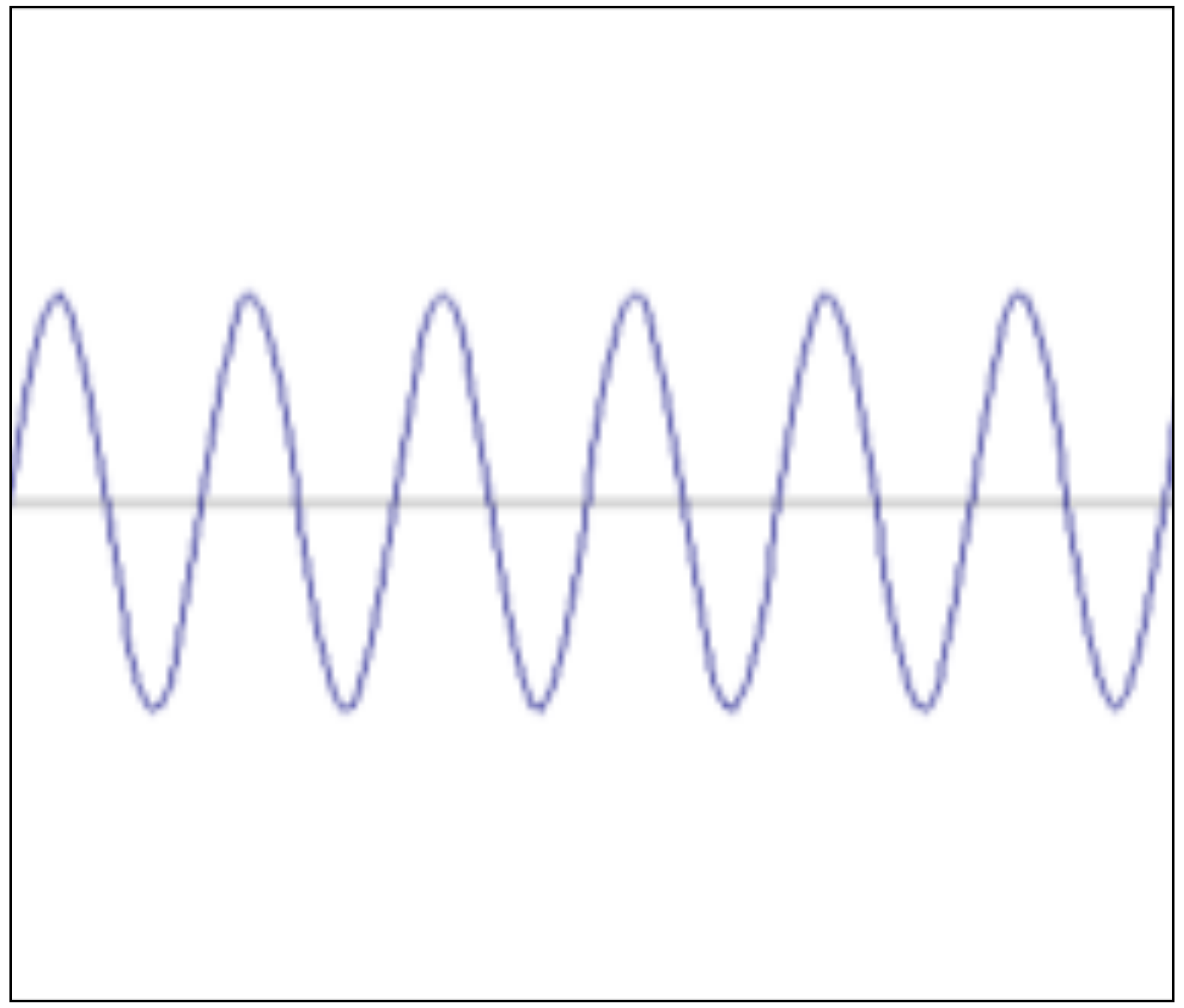


=



$\sin(2\pi x)$

+

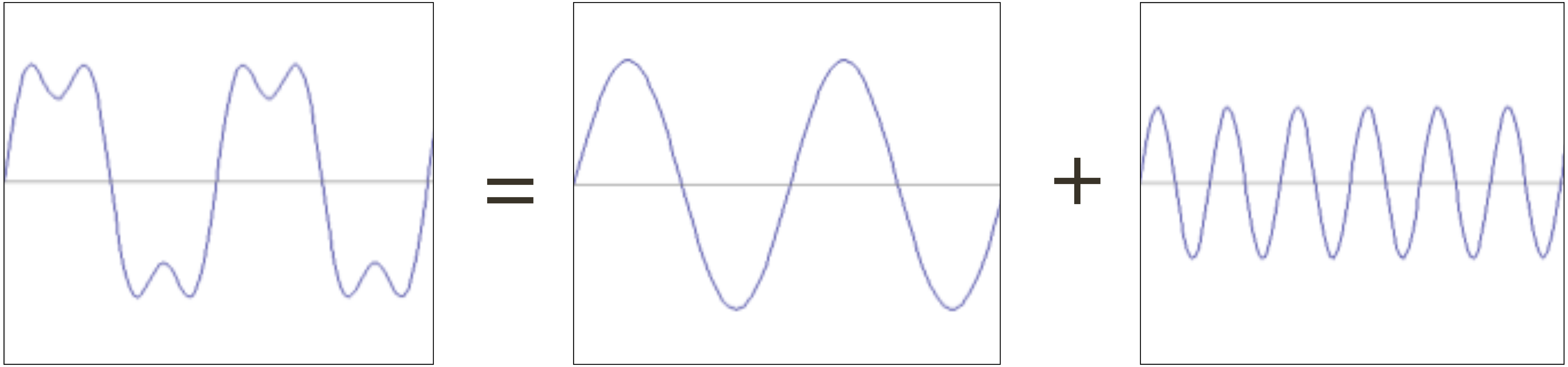


$\frac{1}{3}\sin(2\pi 3x)$

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

Fourier Transform (you will **NOT** be tested on this)

How would you generate this function?



$$f(x) = \sin(2\pi x) + \frac{1}{3}\sin(2\pi 3x)$$

$$\sin(2\pi x)$$

$$\frac{1}{3}\sin(2\pi 3x)$$

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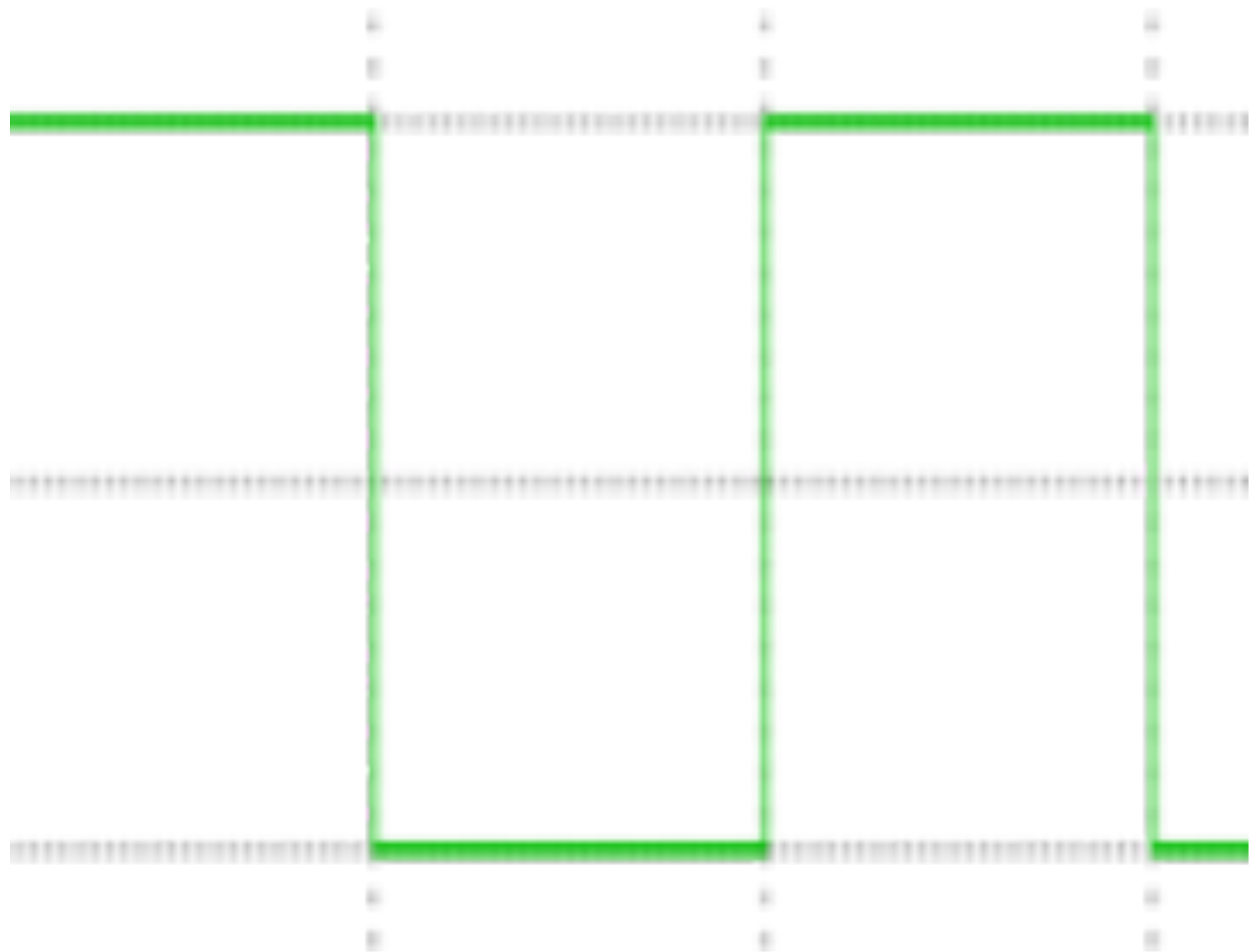


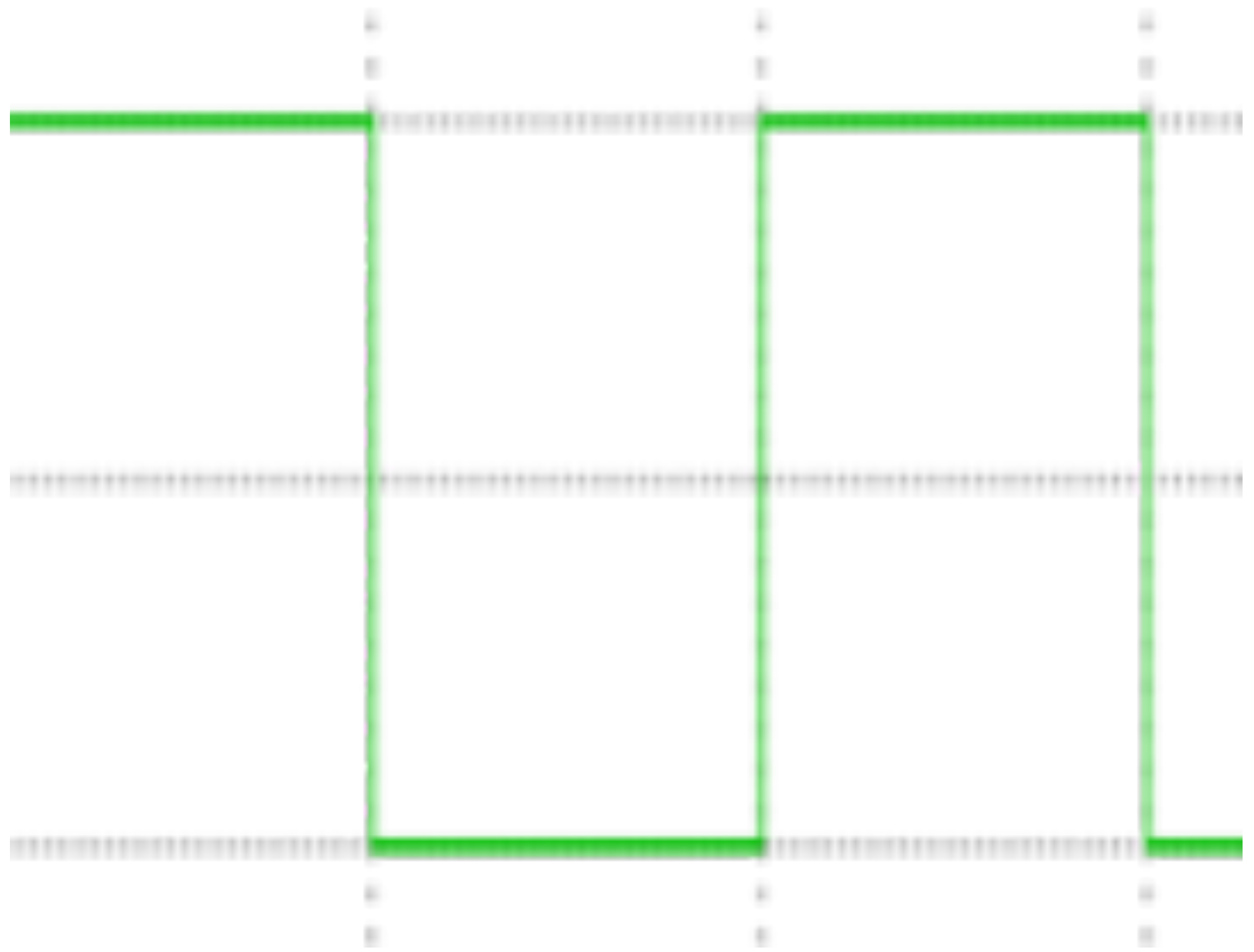
square wave

$\approx$? + ?

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

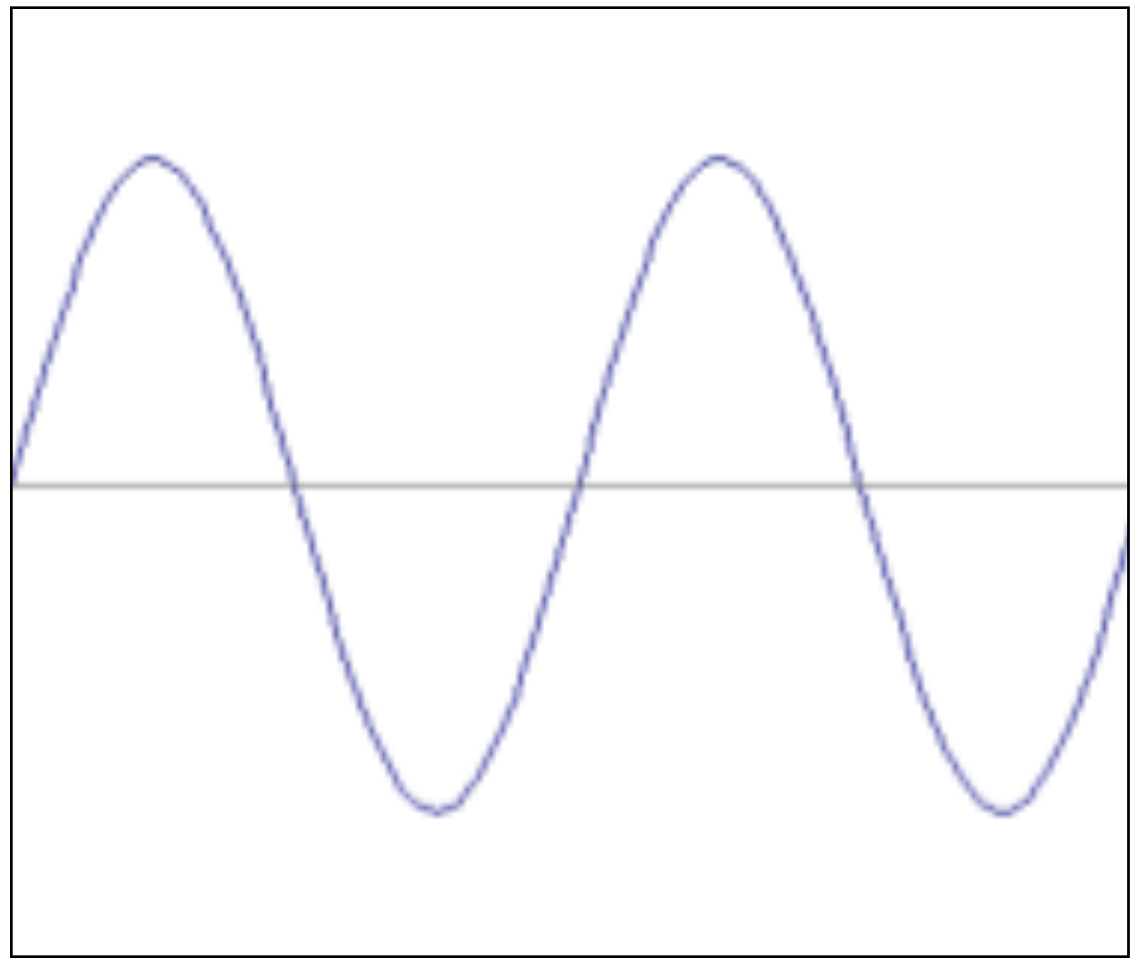
Fourier Transform (you will **NOT** be tested on this)

How would you generate this function?

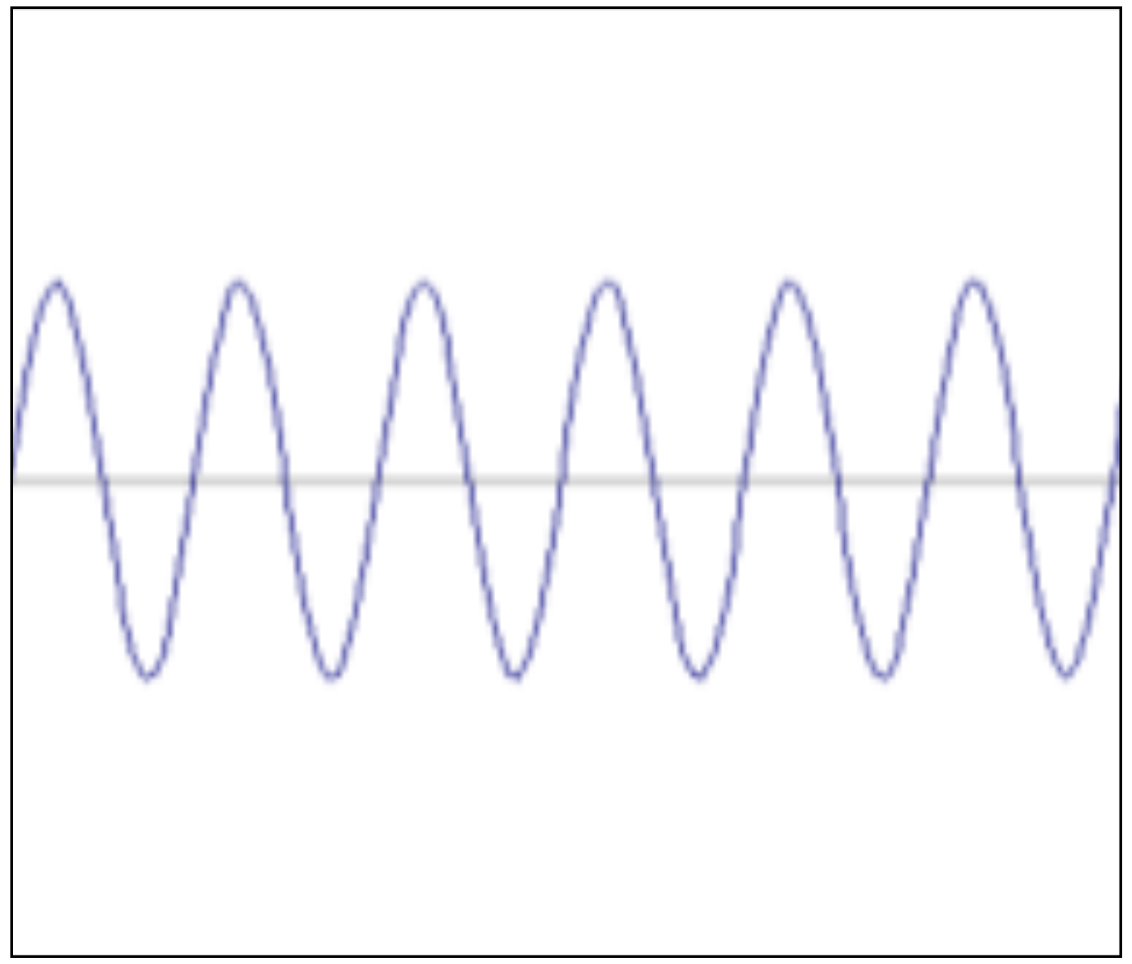


square wave

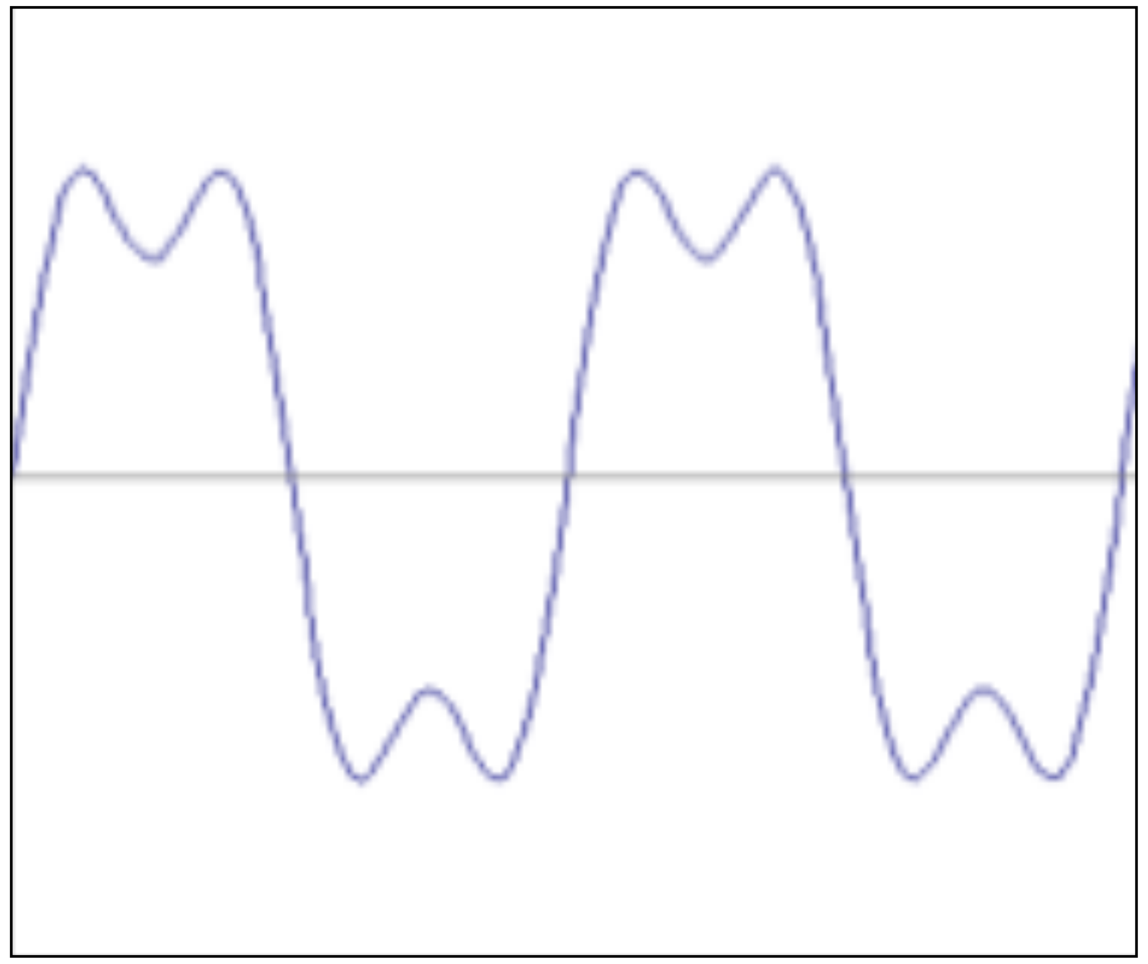
$\approx$



$+$



$=$



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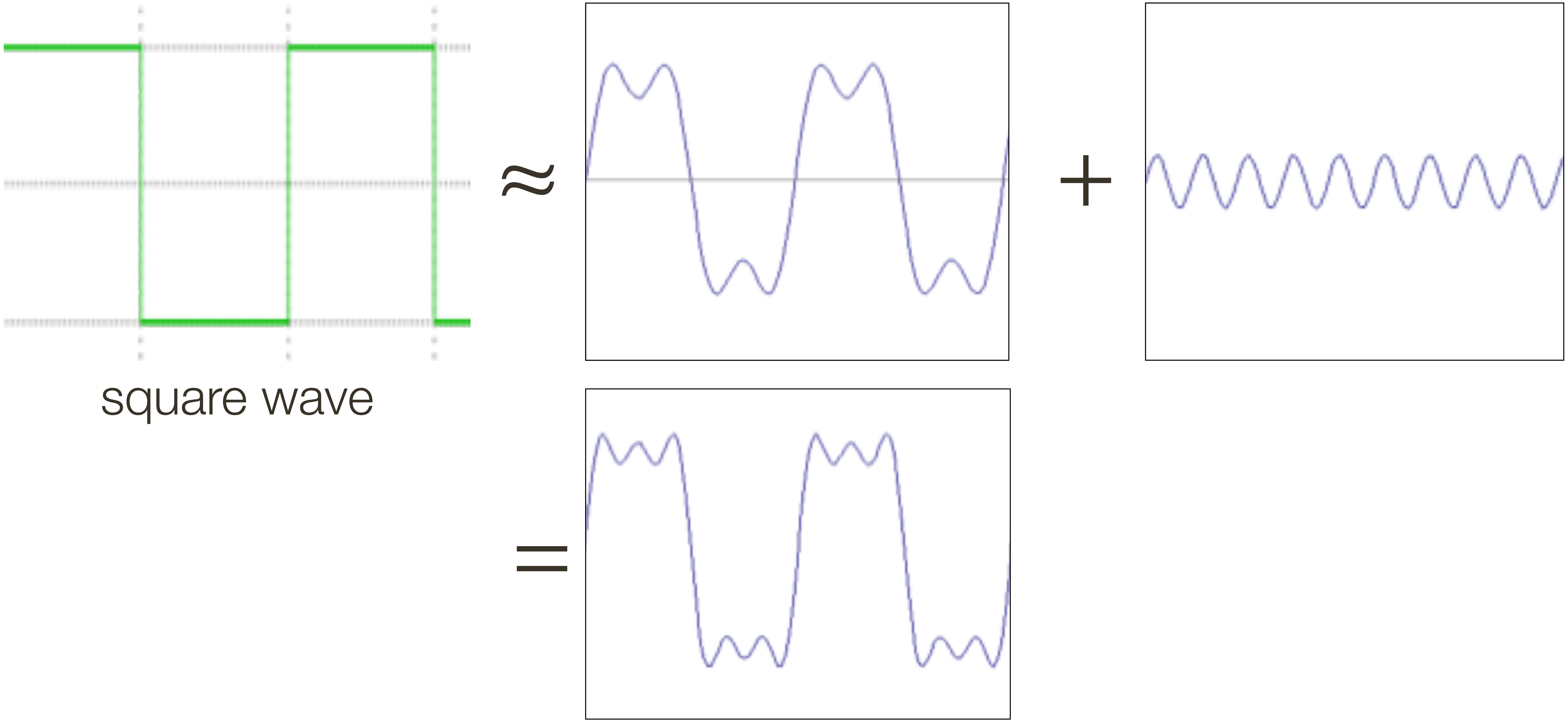

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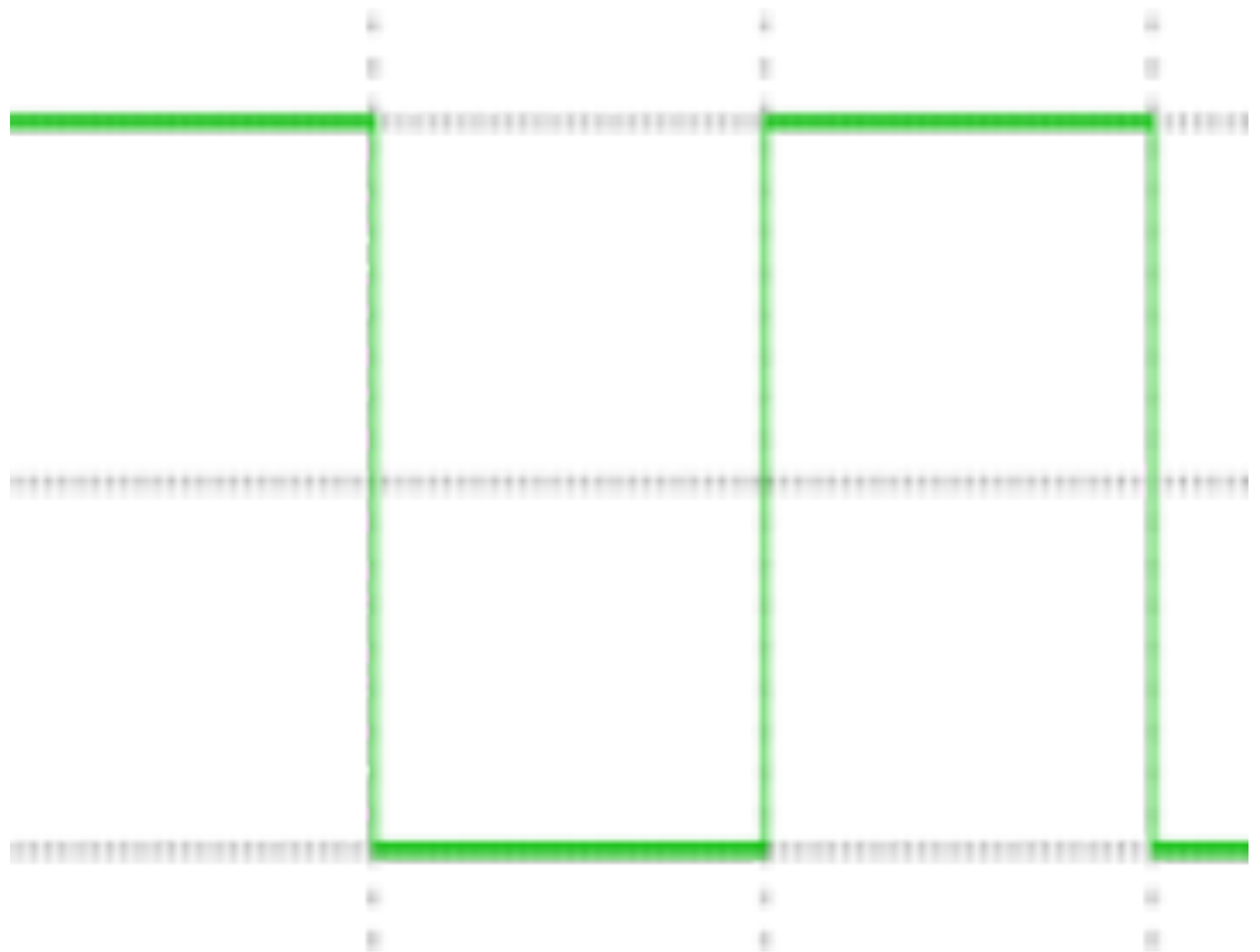
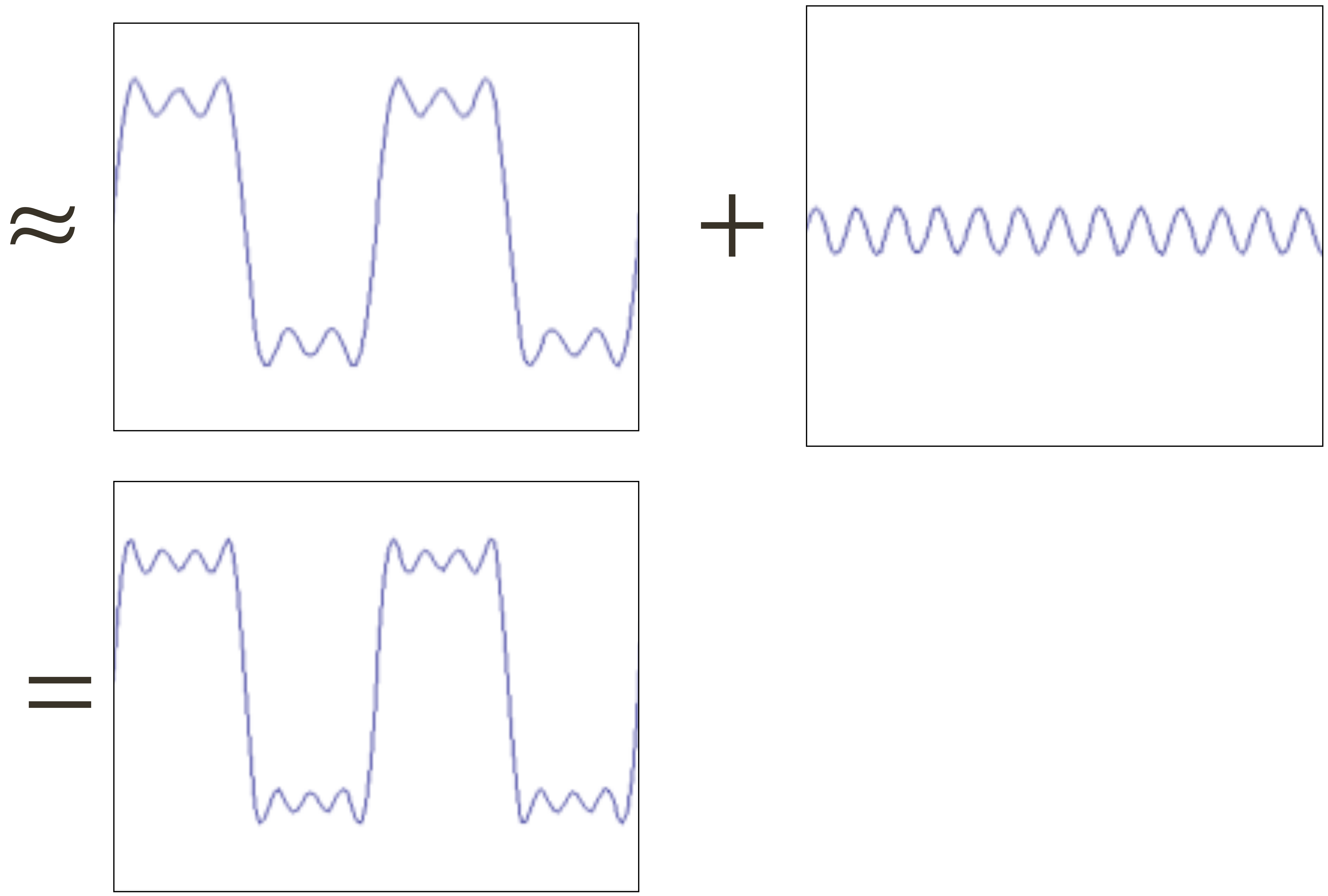


square wave



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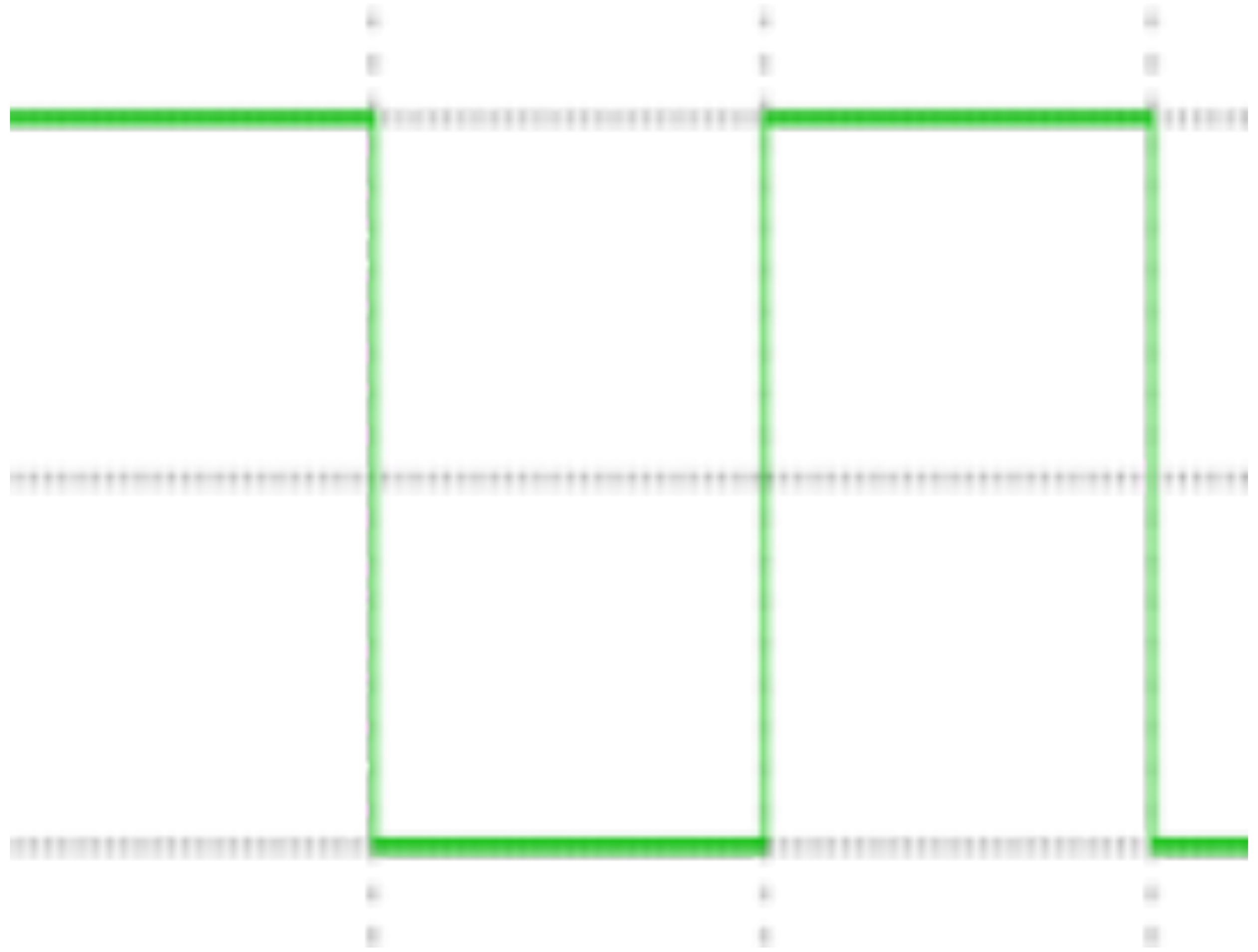

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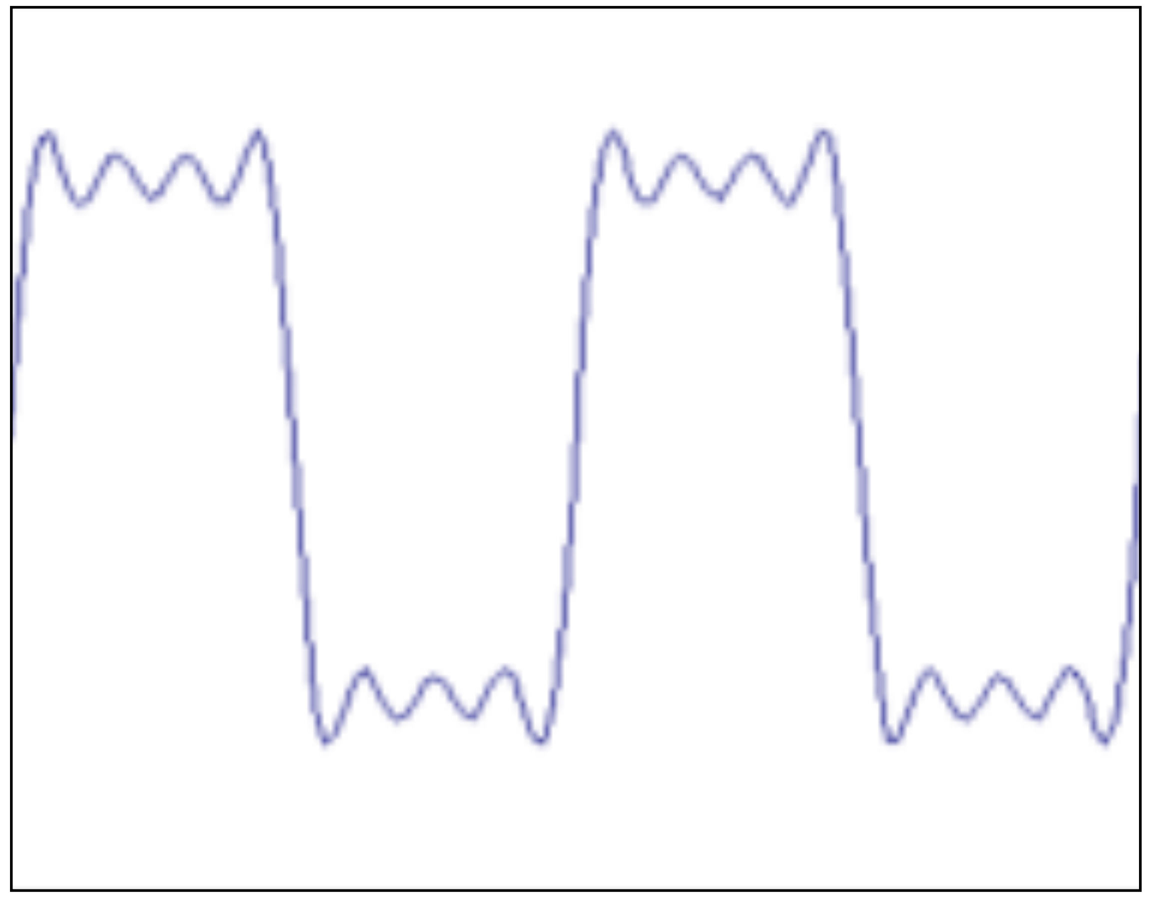
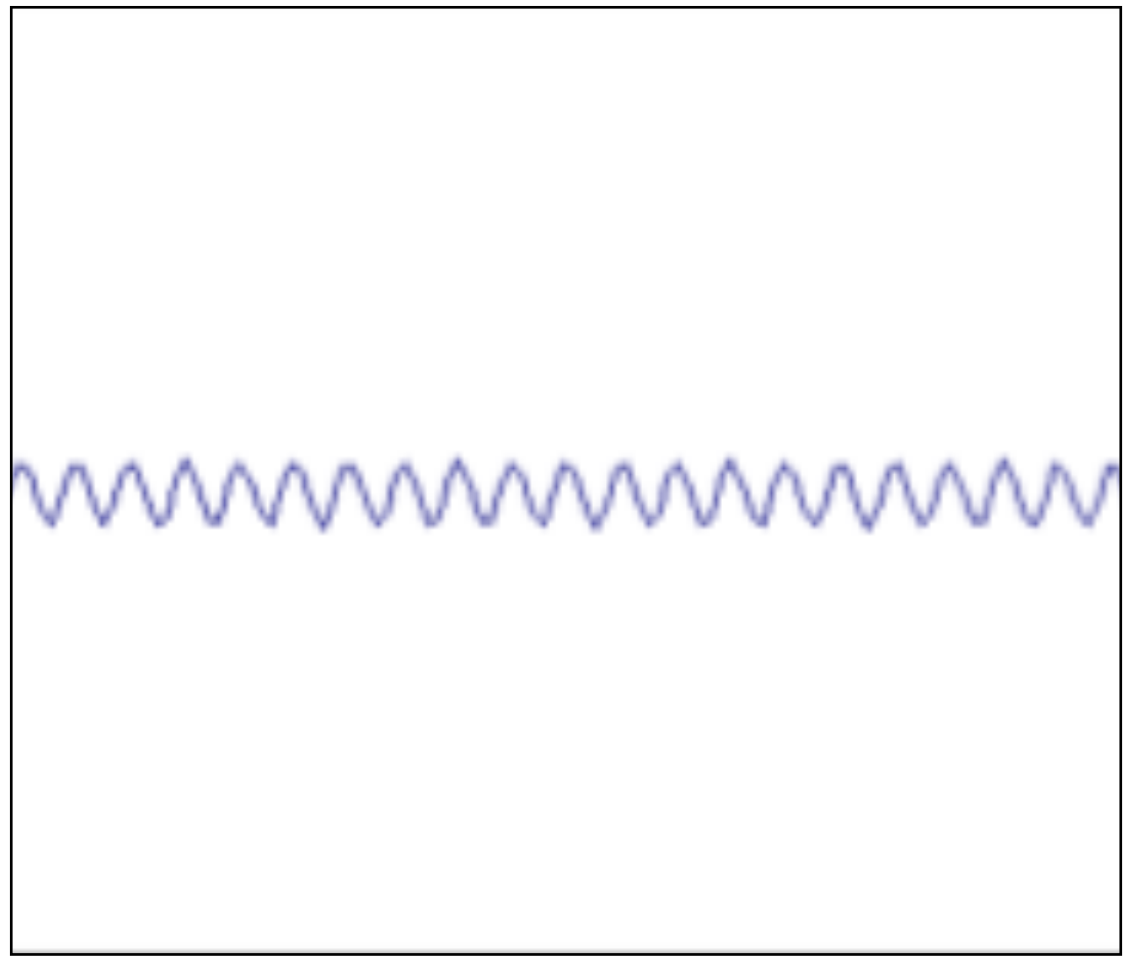
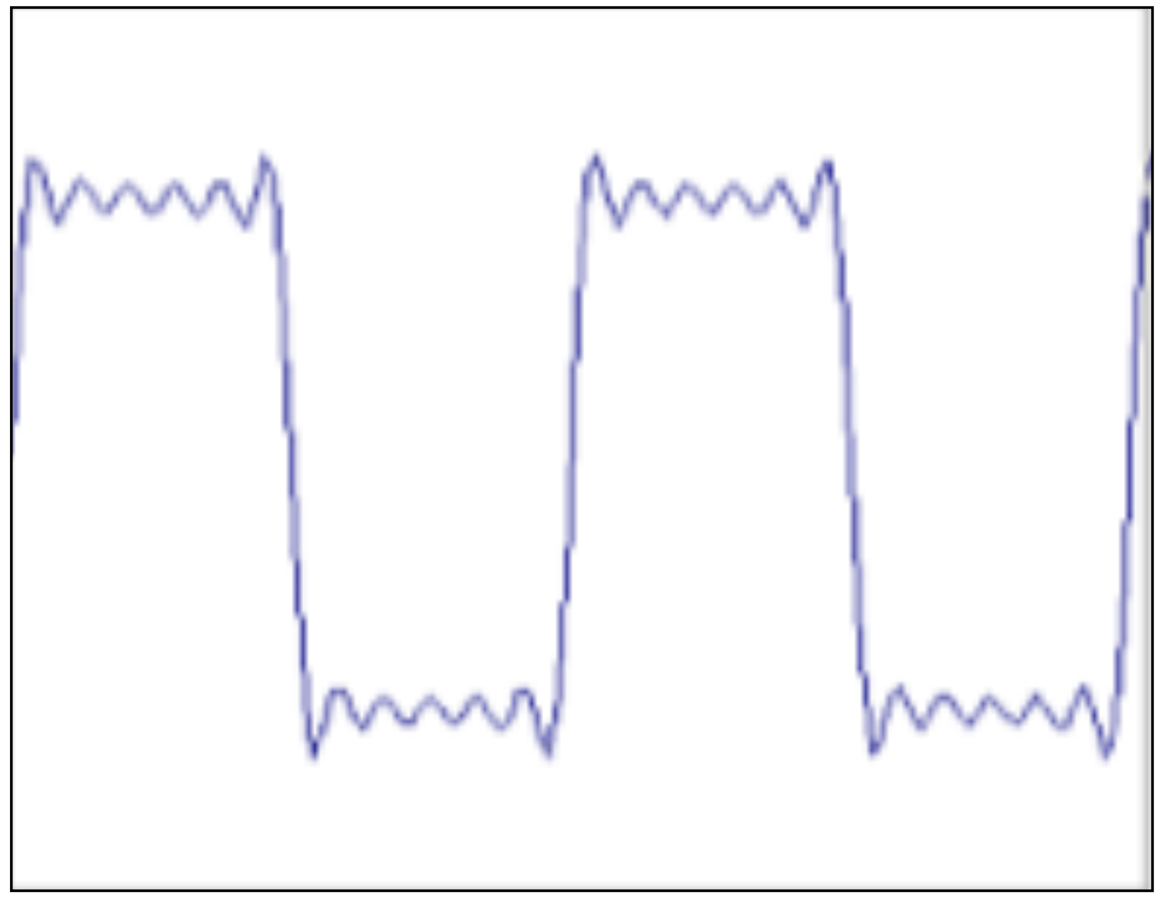
$\approx$



$+$



$=$

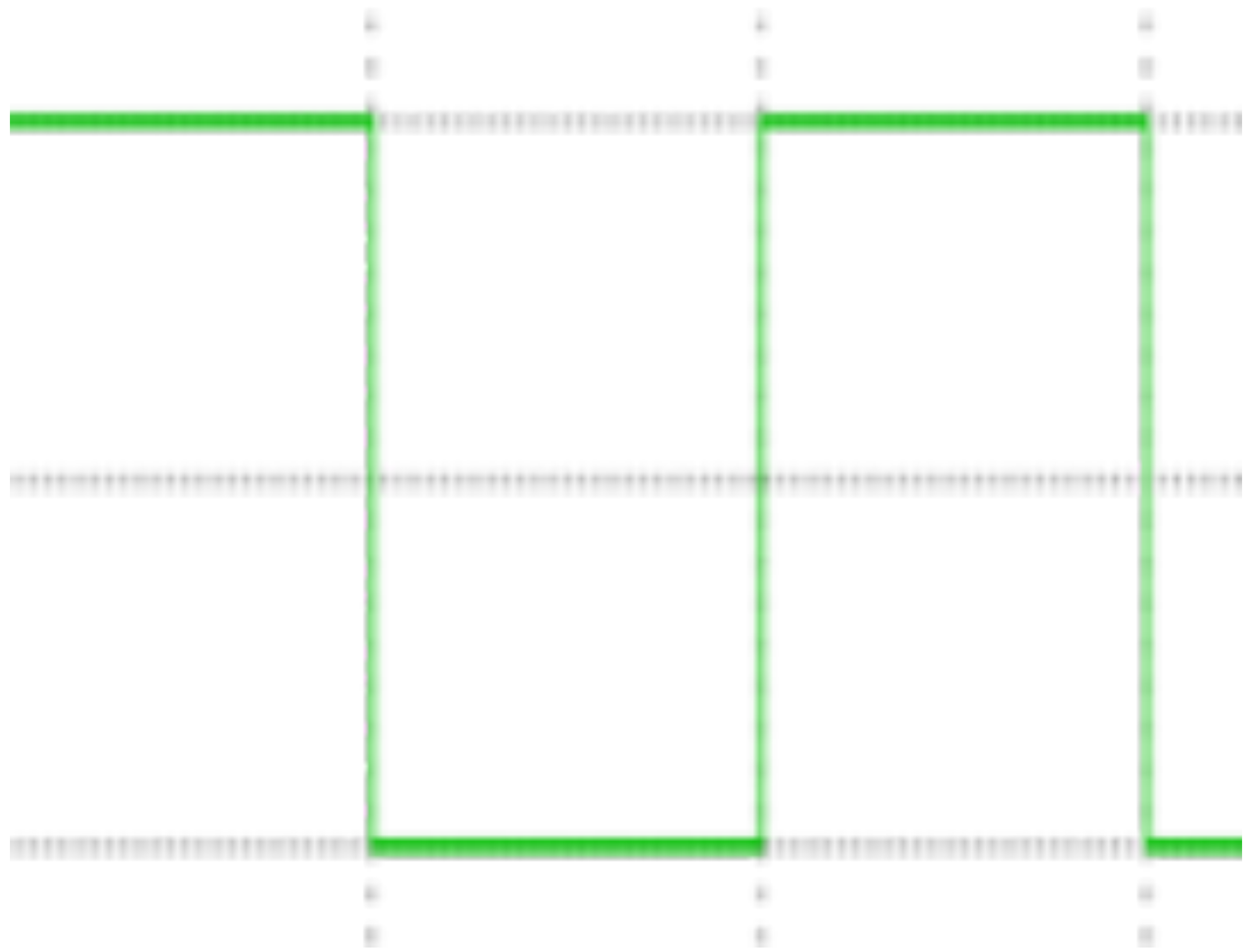


How would you express this mathematically?

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

Fourier Transform (you will **NOT** be tested on this)

How would you generate this function?



square wave

$$= \quad A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kx)$$

infinite sum of sine waves

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

Fourier Transform (you will **NOT** be tested on this)

Basic building block:

$$A \sin(\omega x + \phi)$$

Fourier's claim: Add enough of these to get <u>any</u> periodic signal you want!

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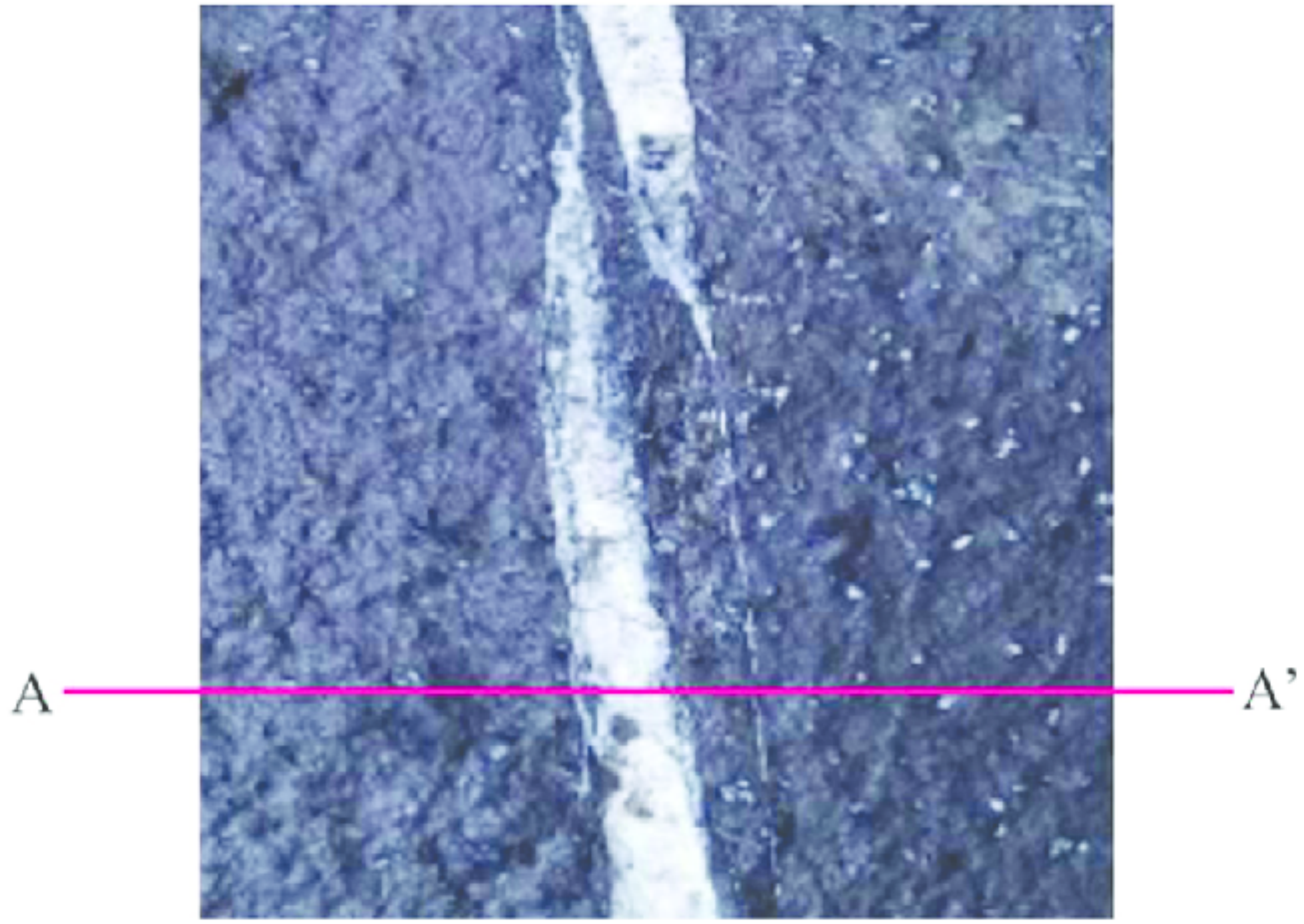

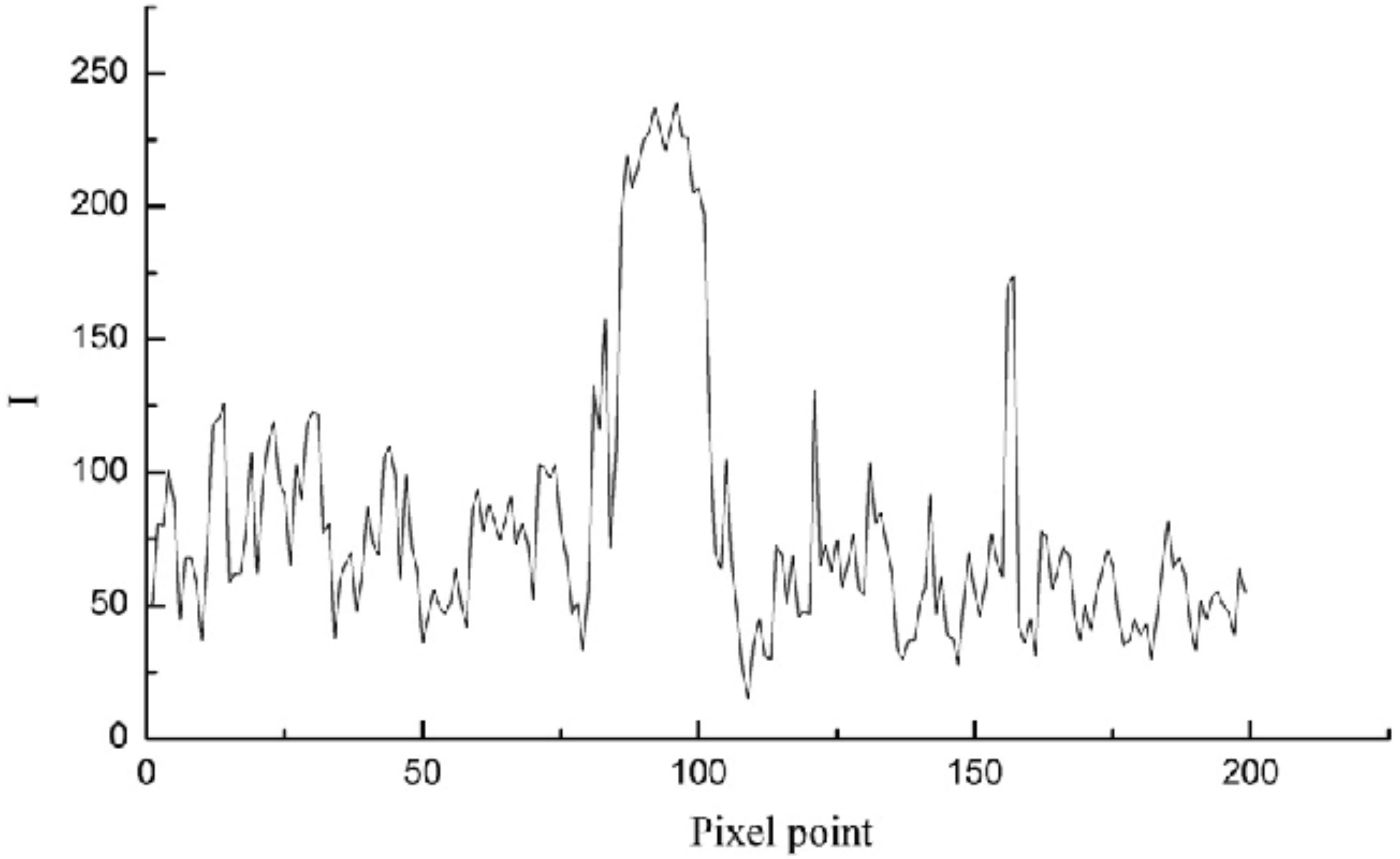

Image from: Numerical Simulation and Fractal Analysis of Mesoscopic Scale Failure in Shale Using Digital Images

Fourier Transform (you will **NOT** be tested on this)

What are “frequencies” in an image?

Spatial frequency

$f = 4$ $f = 5$ $f = 6$ $f = 7$ $f = 8$ $f = 9$ $f = 10$

Fourier Transform (you will **NOT** be tested on this)

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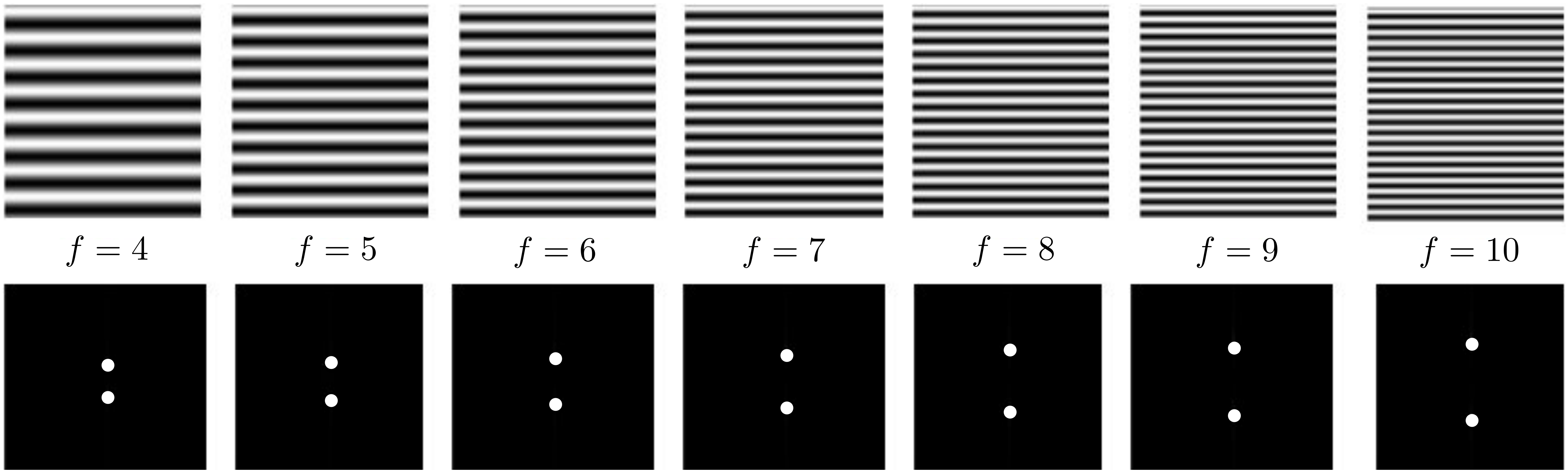
Spatial frequency



Amplitude (magnitude) of Fourier transform (phase does not show desirable correlations with image structure)

Fourier Transform (you will **NOT** be tested on this)

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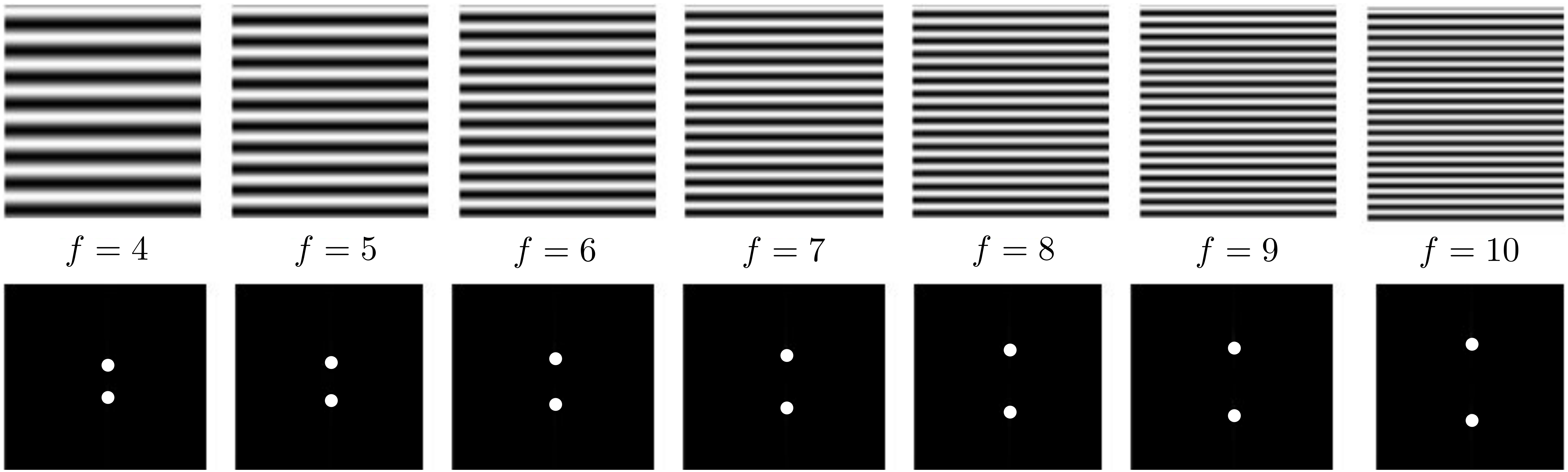


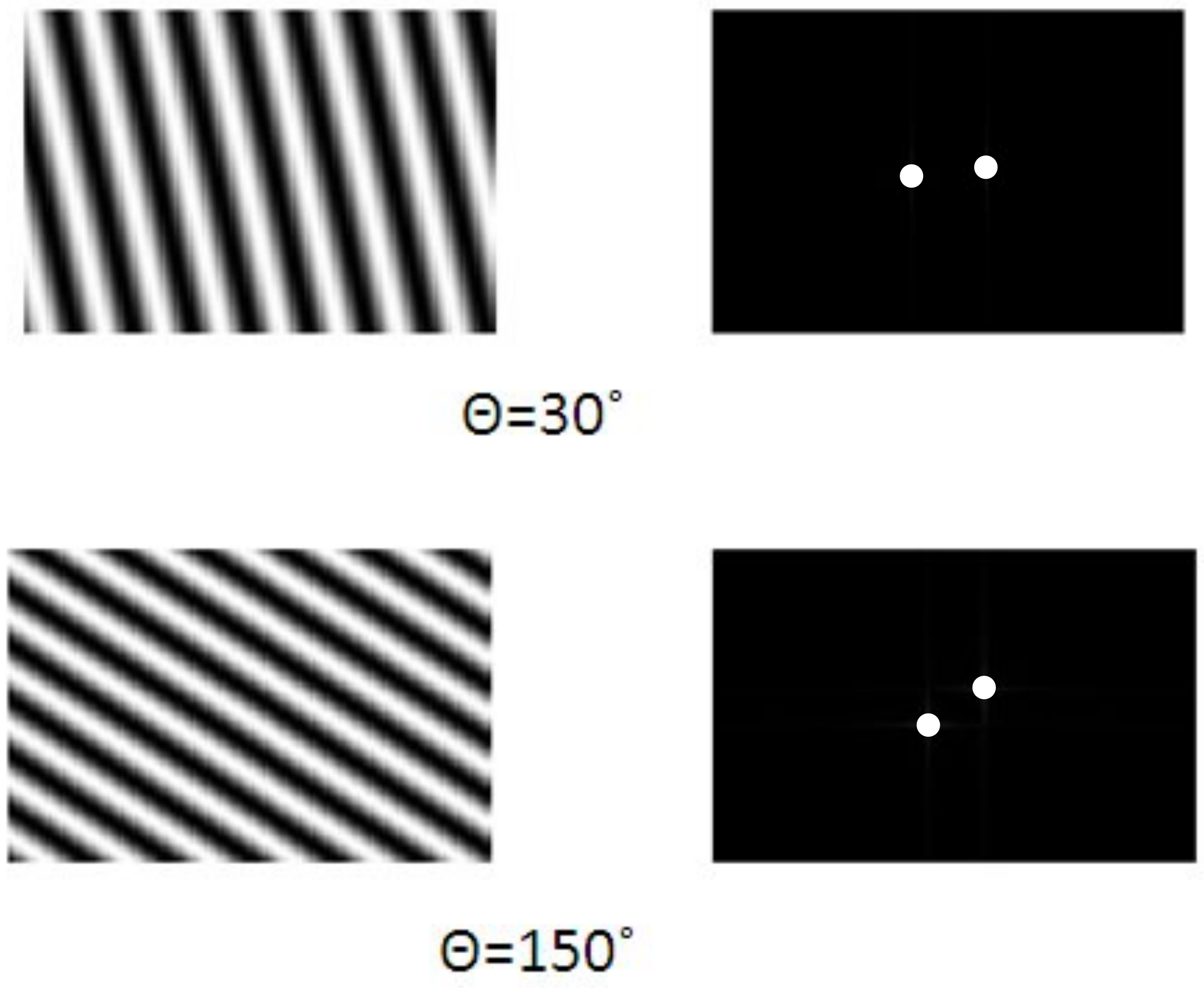
Amplitude (magnitude) of Fourier transform (phase does not show desirable correlations with image structure)

Observation: low frequencies close to the center

Fourier Transform (you will **NOT** be tested on this)

What are “frequencies” in an image?

Spatial frequency



Fourier Transform (you will **NOT** be tested on this)

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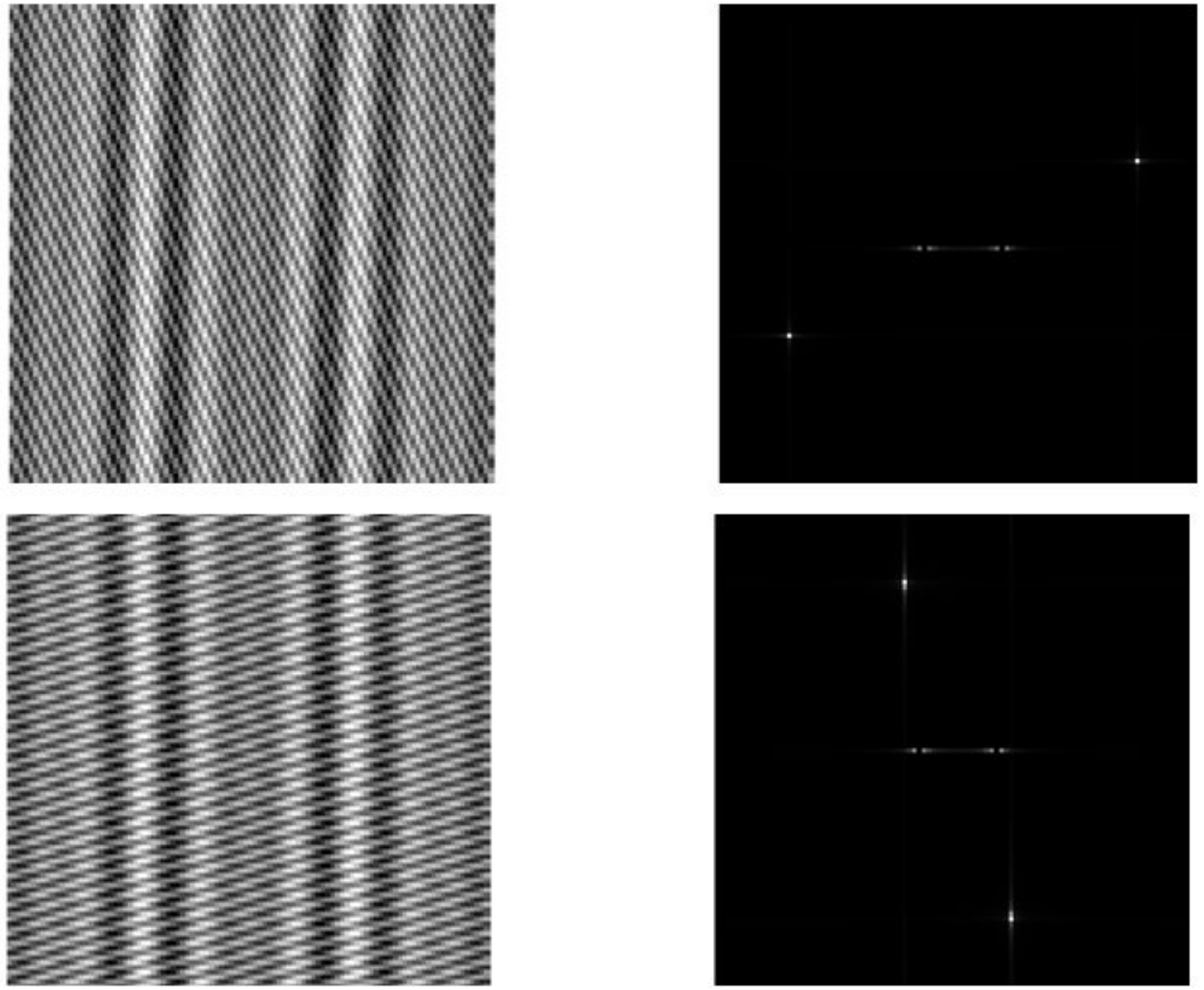


Fourier Transform (you will **NOT** be tested on this)

Image

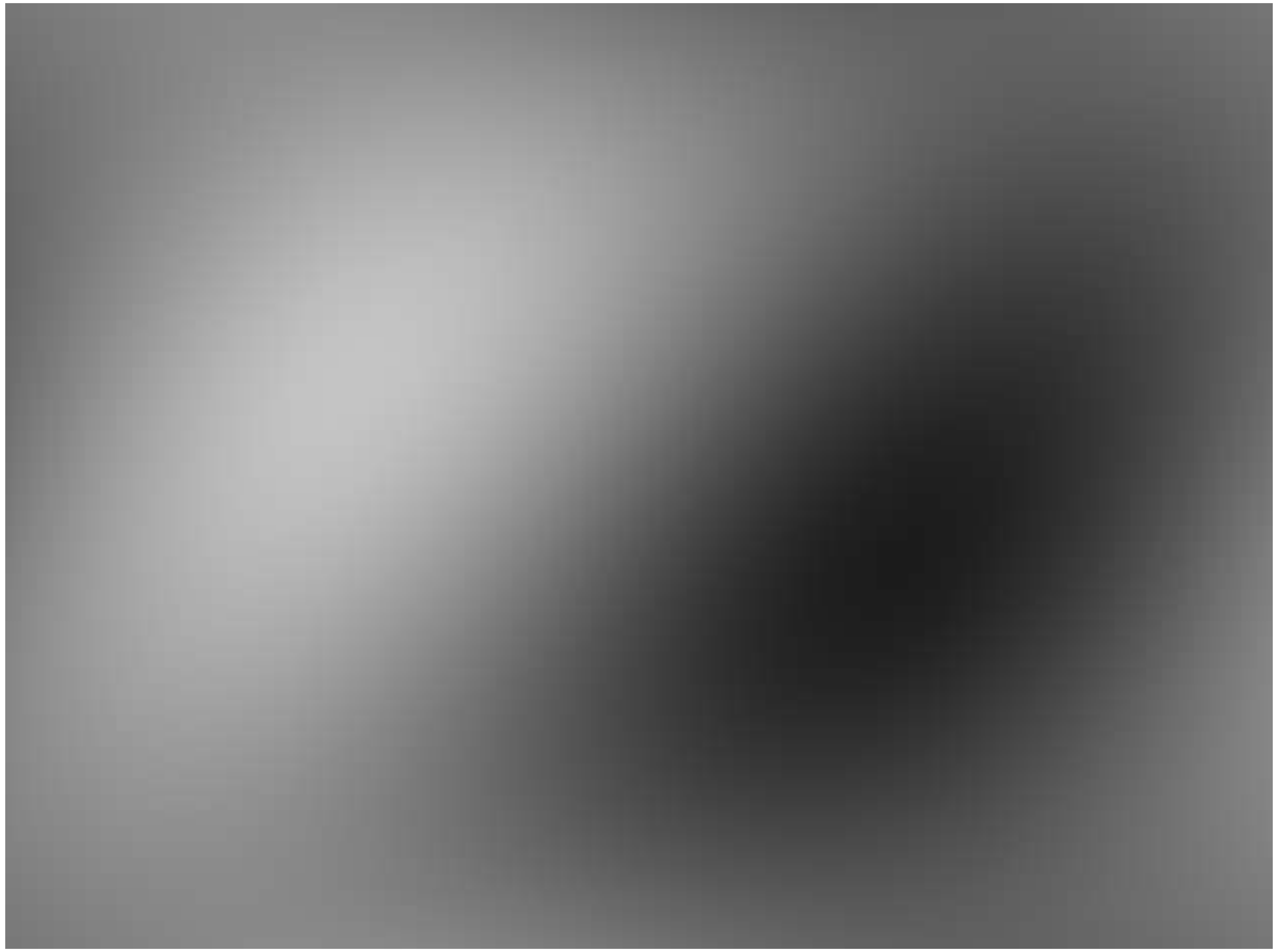
https://photo.stackexchange.com/questions/40401/what-does-frequency-mean-in-an-image/40410#40410

Fourier Transform (you will **NOT** be tested on this)

First (lowest) frequency, a.k.a. average

https://photo.stackexchange.com/questions/40401/what-does-frequency-mean-in-an-image/40410#40410

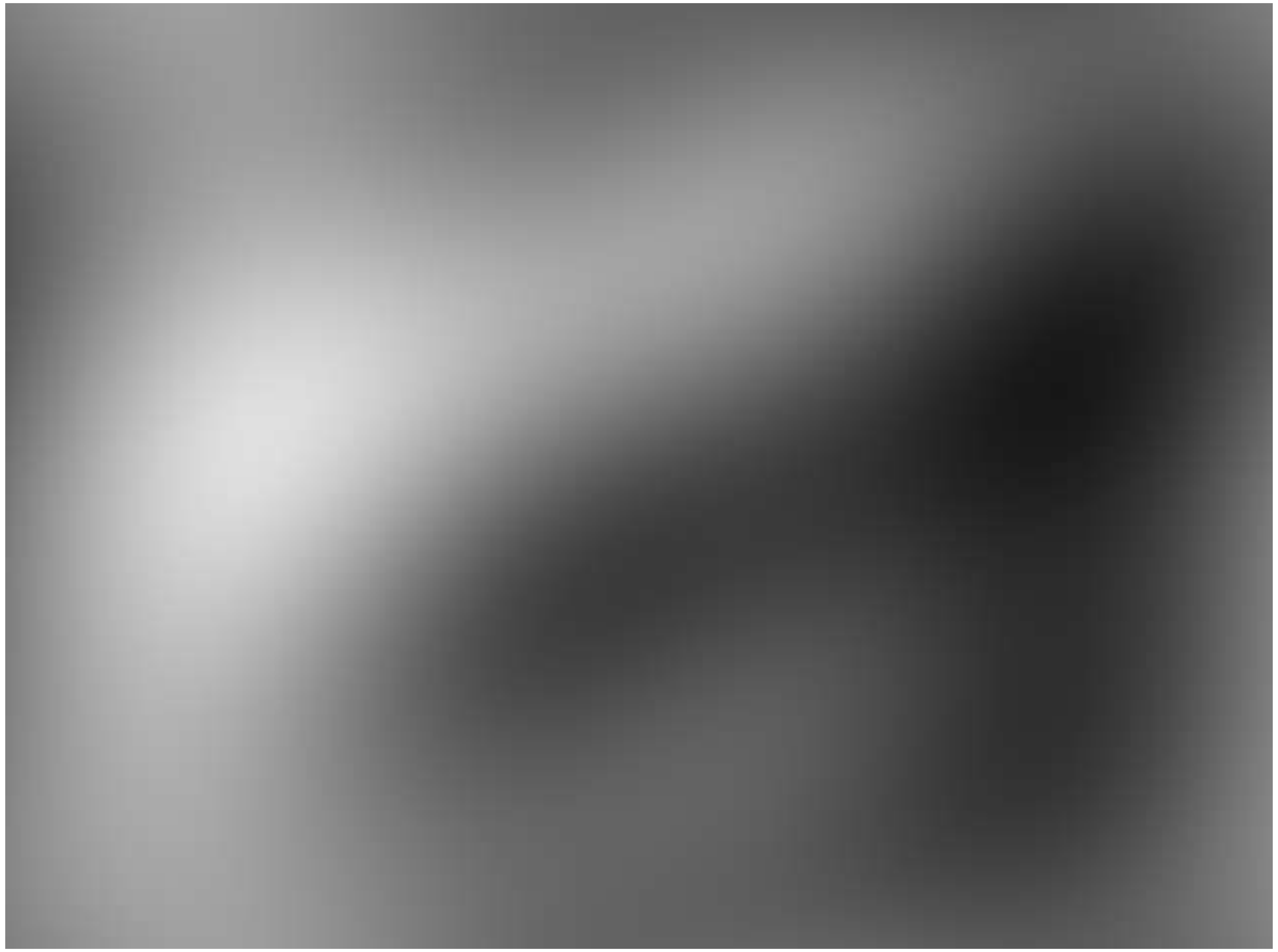
Fourier Transform (you will **NOT** be tested on this)



+ **Second** frequency

https://photo.stackexchange.com/questions/40401/what-does-frequency-mean-in-an-image/40410#40410

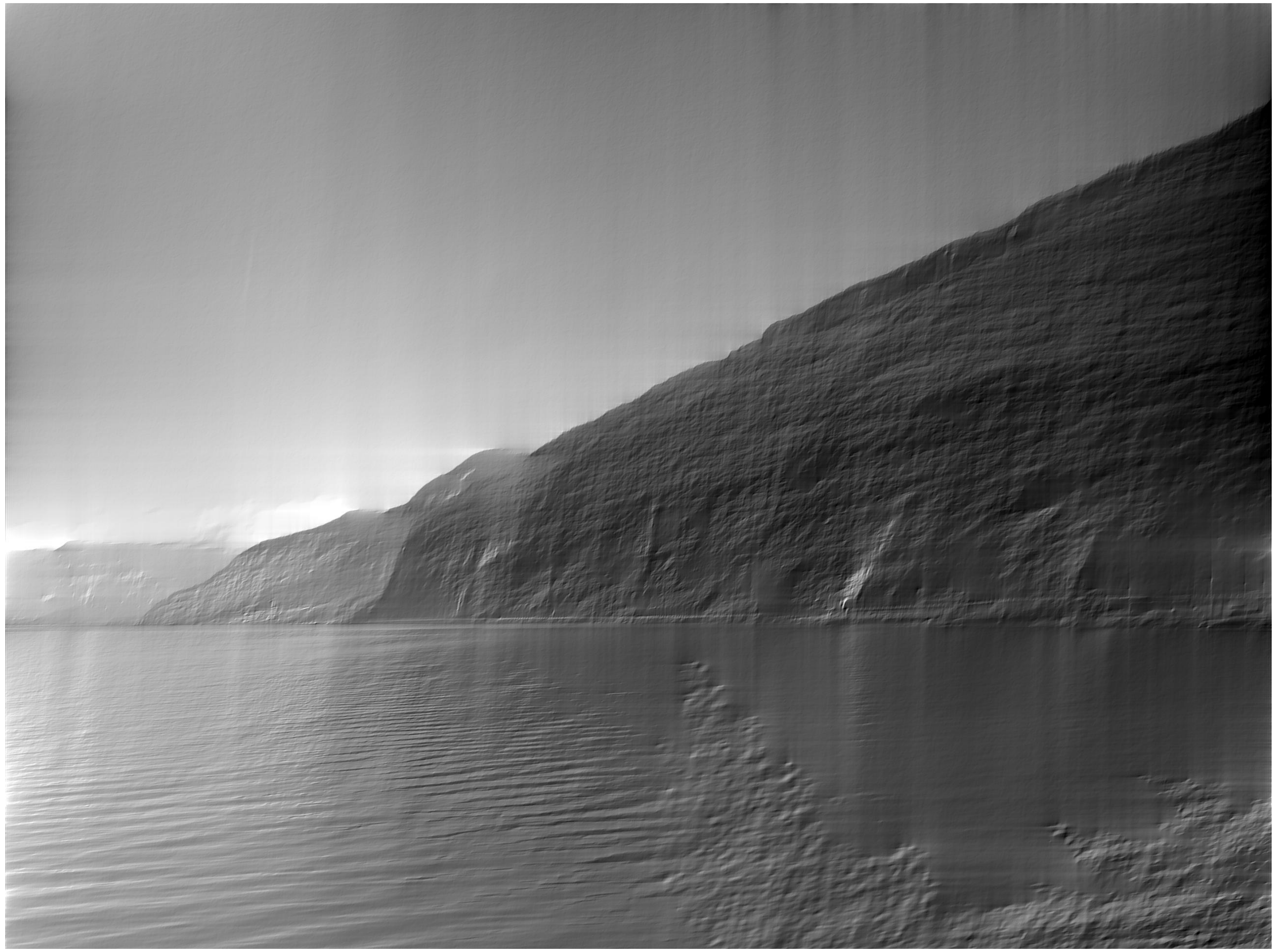
Fourier Transform (you will **NOT** be tested on this)



+ **Third** frequency

https://photo.stackexchange.com/questions/40401/what-does-frequency-mean-in-an-image/40410#40410

Fourier Transform (you will **NOT** be tested on this)



\+ **50%** of frequencies

https://photo.stackexchange.com/questions/40401/what-does-frequency-mean-in-an-image/40410#40410

Fourier Transform (you will **NOT** be tested on this)

https://photo.stackexchange.com/questions/40401/what-does-frequency-mean-in-an-image/40410#40410

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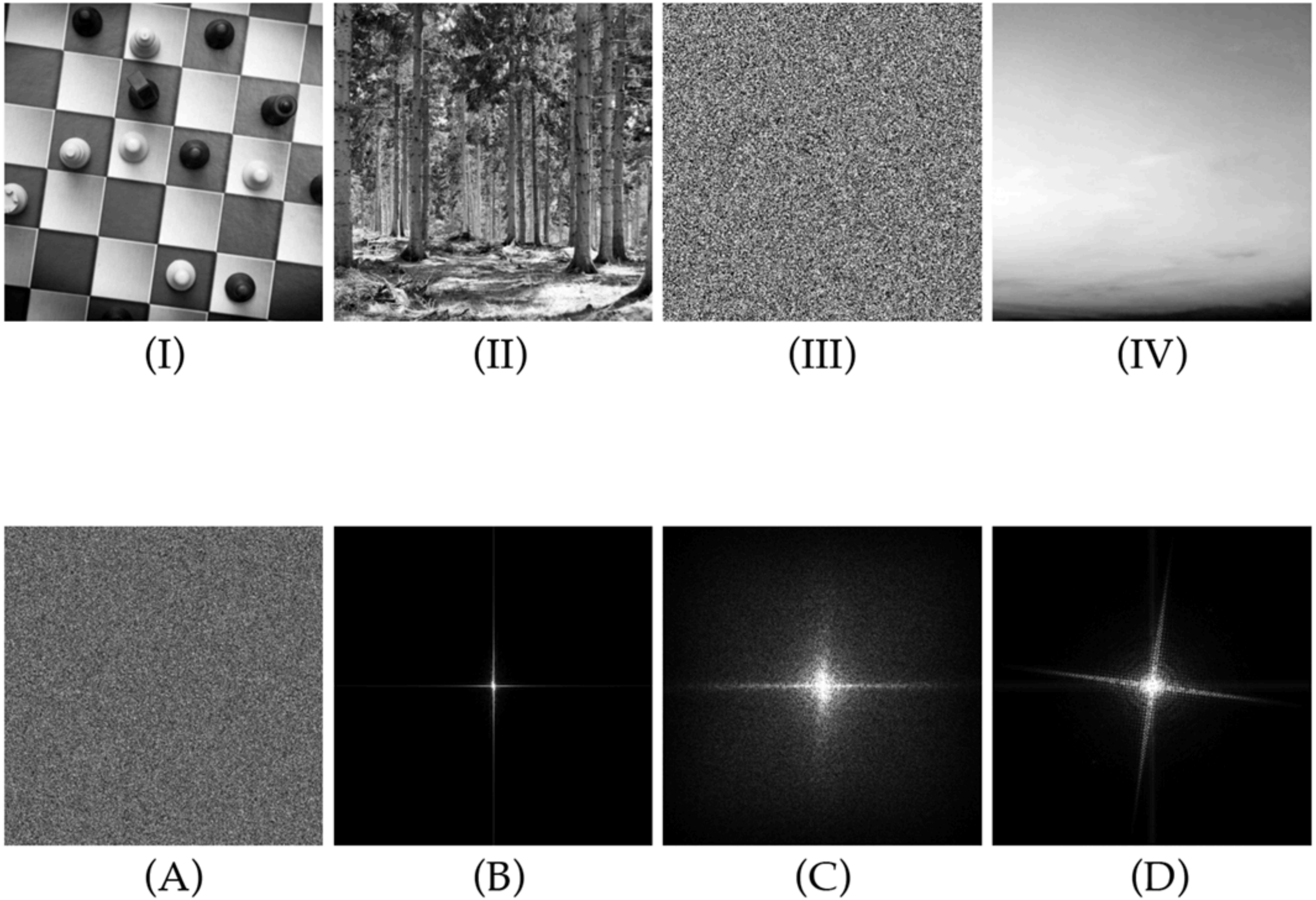

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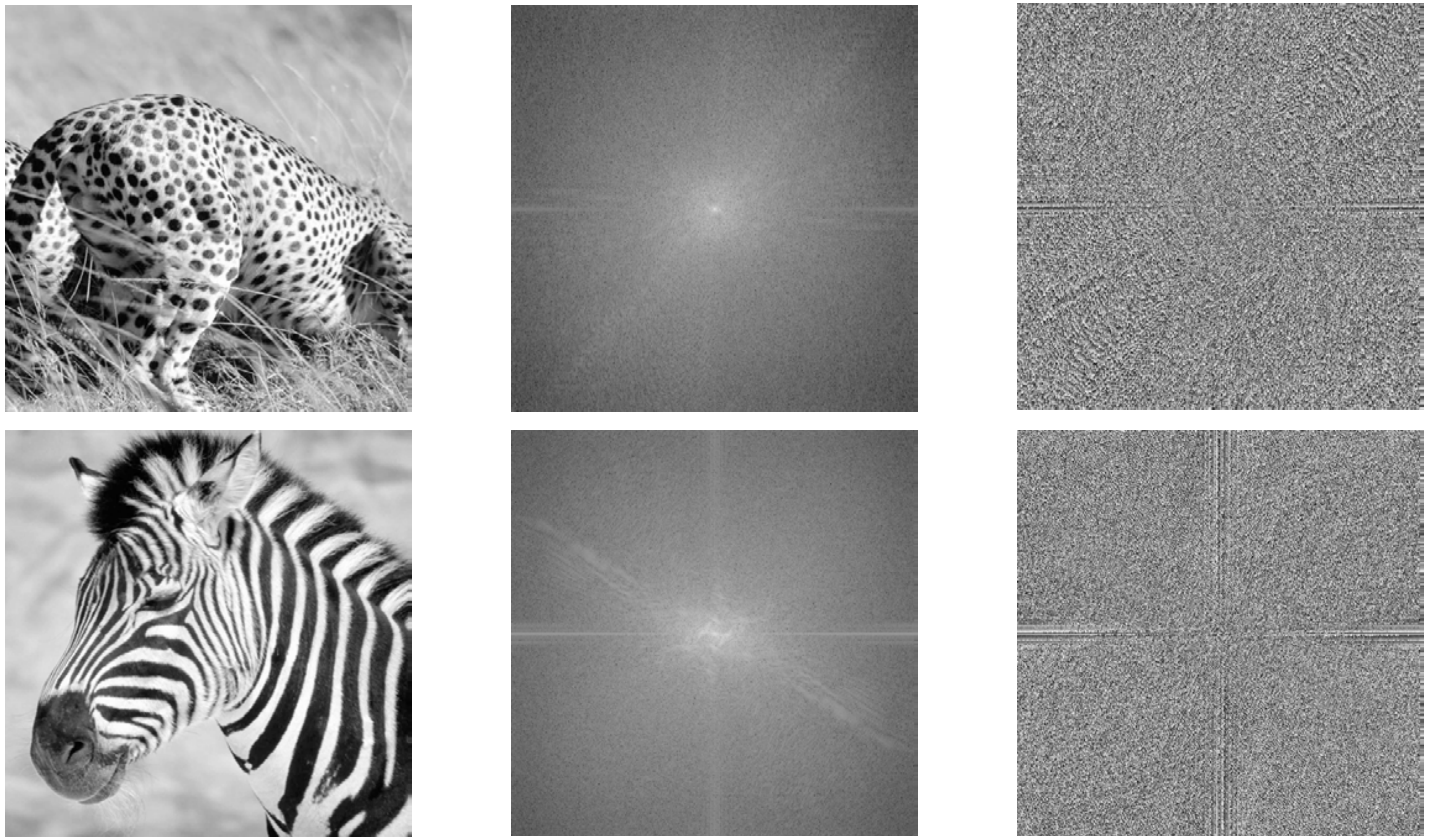
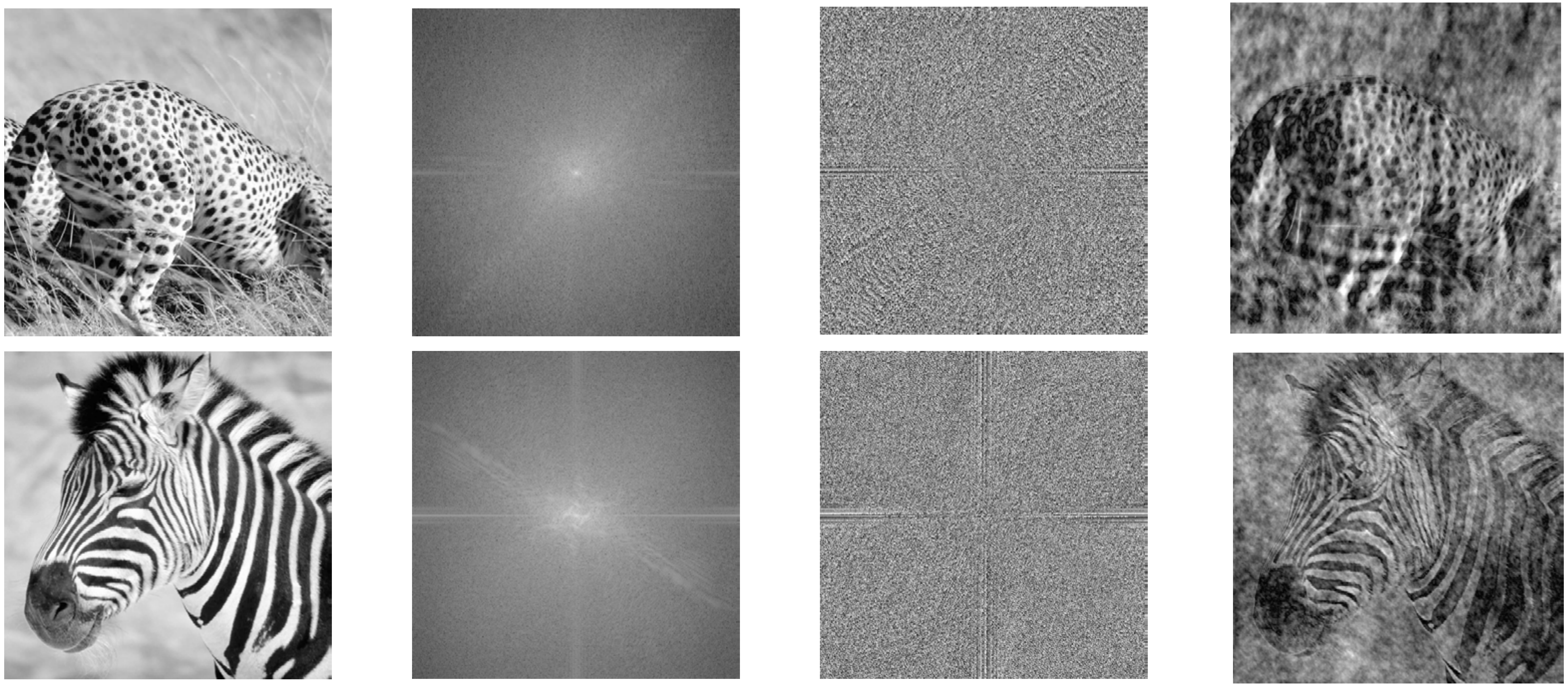


amplitude phase

Forsyth & Ponce (2nd ed.) Figure 4.6

Fourier Transform (you will **NOT** be tested on this)



cheetah phase with zebra amplitude

zebra phase with cheetah amplitude

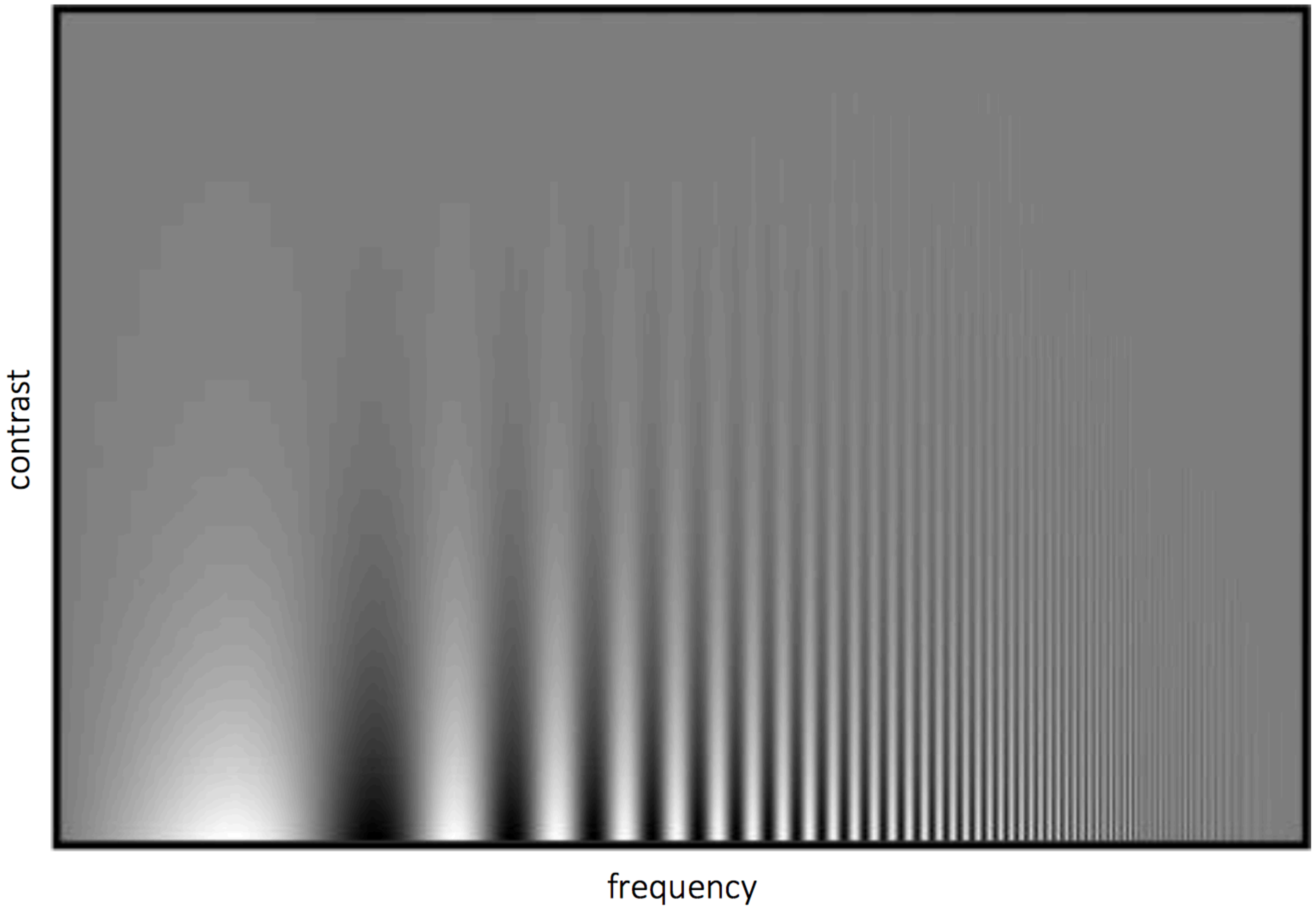
amplitude

phase

Forsyth & Ponce (2nd ed.) Figure 4.6

Fourier Transform (you will **NOT** be tested on this)

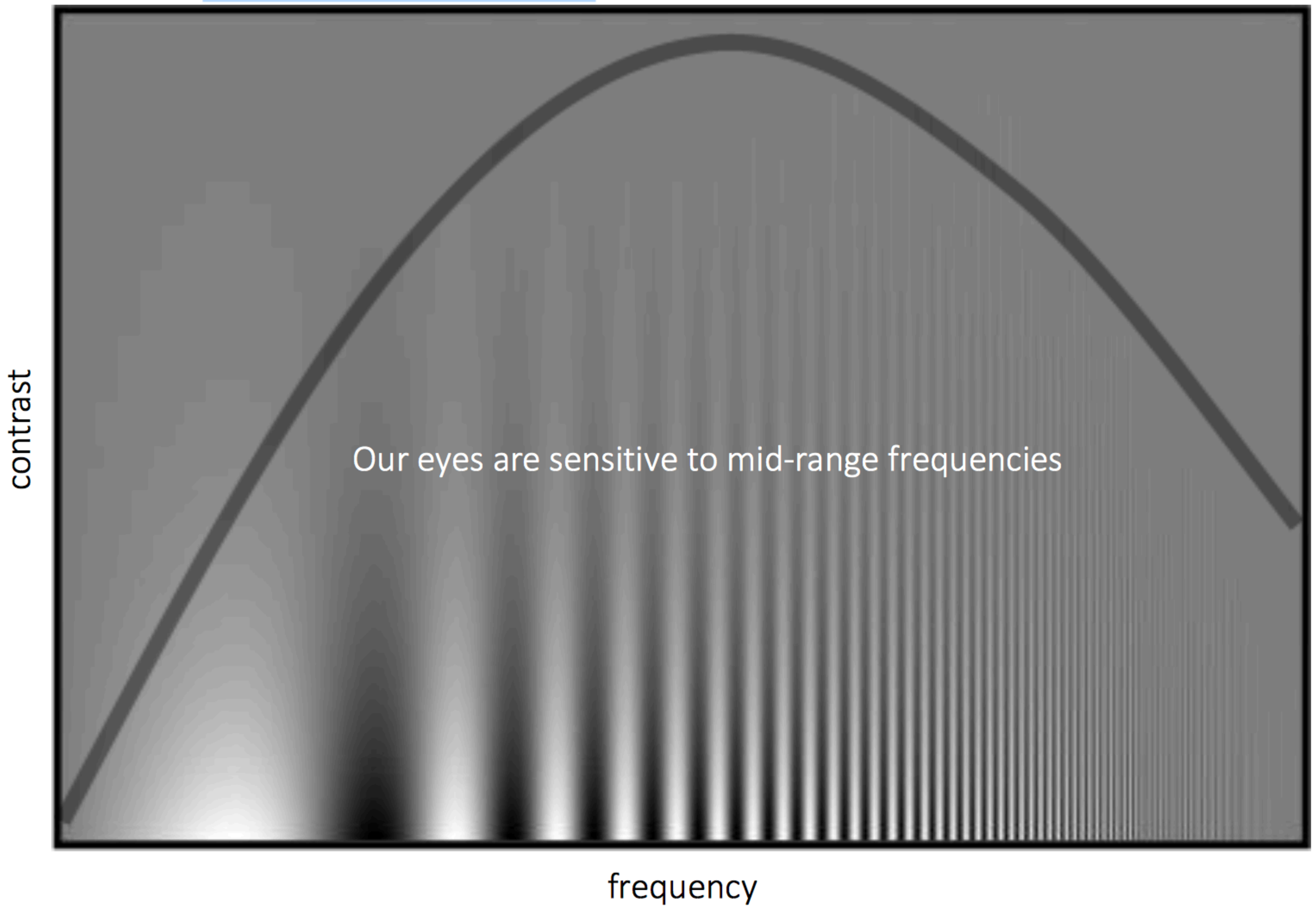
Experiment: Where of you see the stripes?



Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

Fourier Transform (you will **NOT** be tested on this)

Campbell-Robson contrast sensitivity curve



Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

What preceded was for fun

(you will **NOT** be tested on it)

Fourier Transform

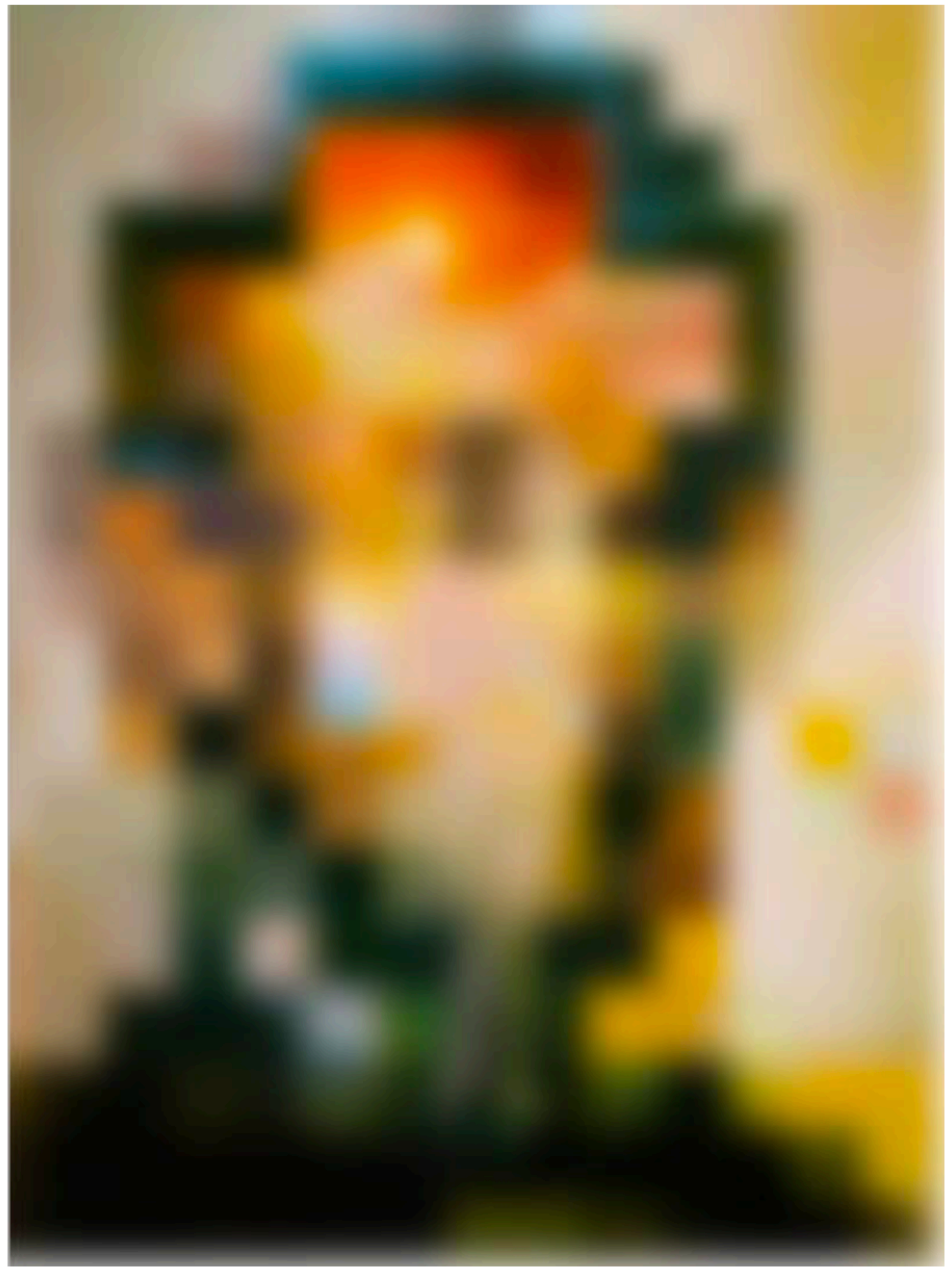
Preview of **Part 3** of your homework

Gala Contemplating the Mediterranean Sea Which at Twenty Meters Becomes the Portrait of Abraham Lincoln (Homage to Rothko)

Salvador Dali, 1976

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

Fourier Transform



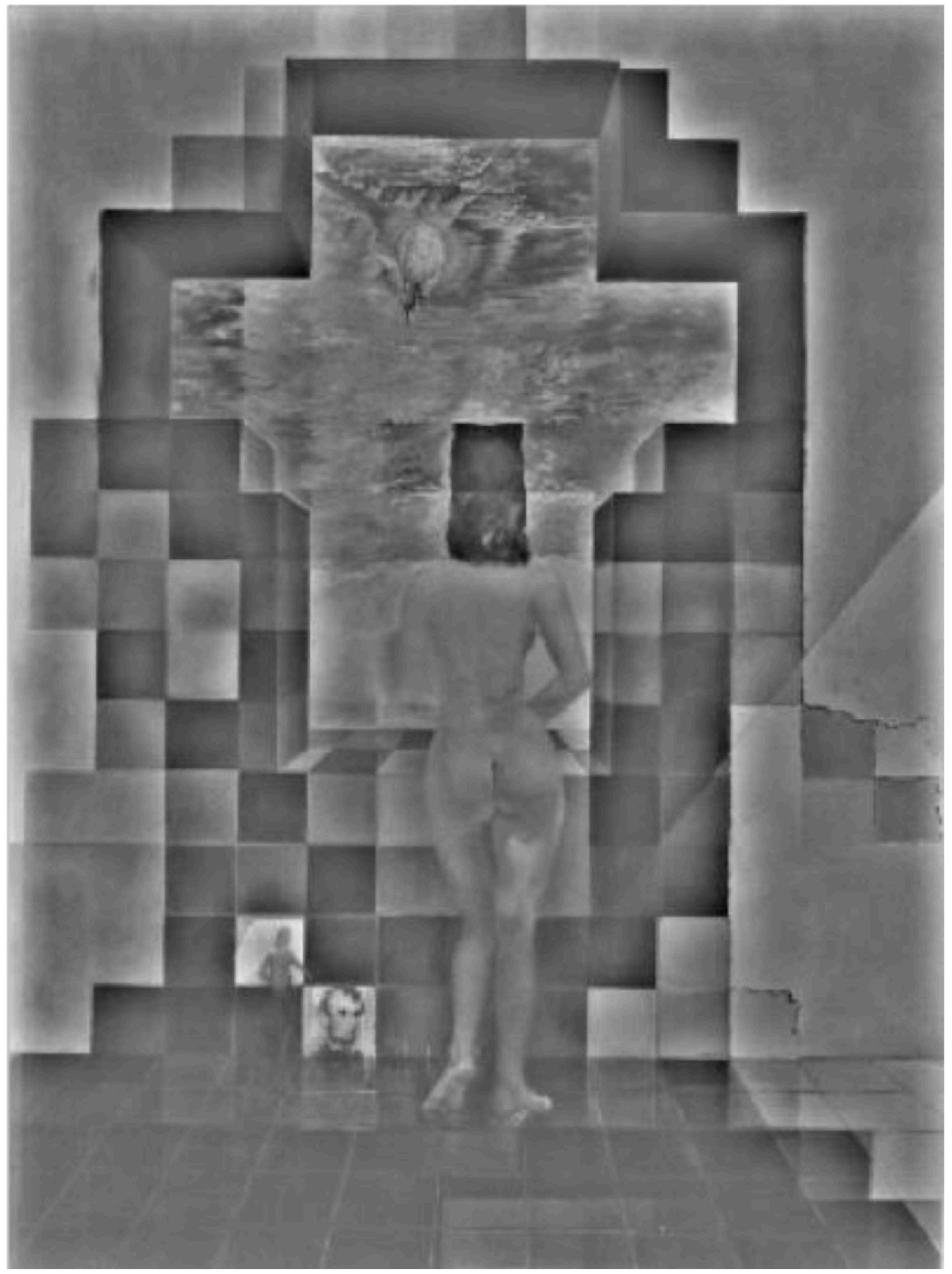
Low-pass filtered version

Preview of **Part 3** of your homework

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

Fourier Transform

Preview of **Part 3** of your homework



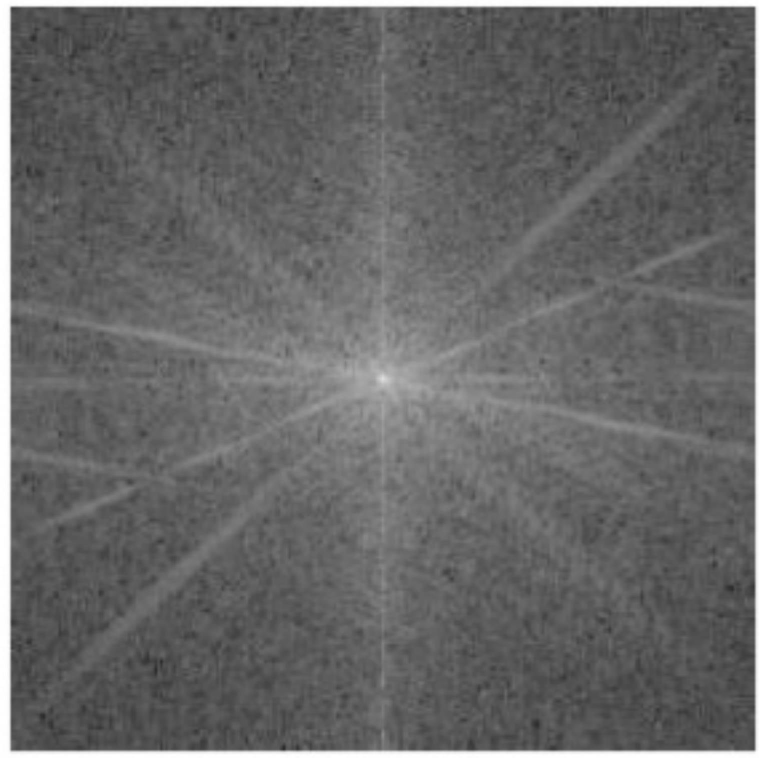
High-pass filtered version

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

Aside: You will not be tested on this …

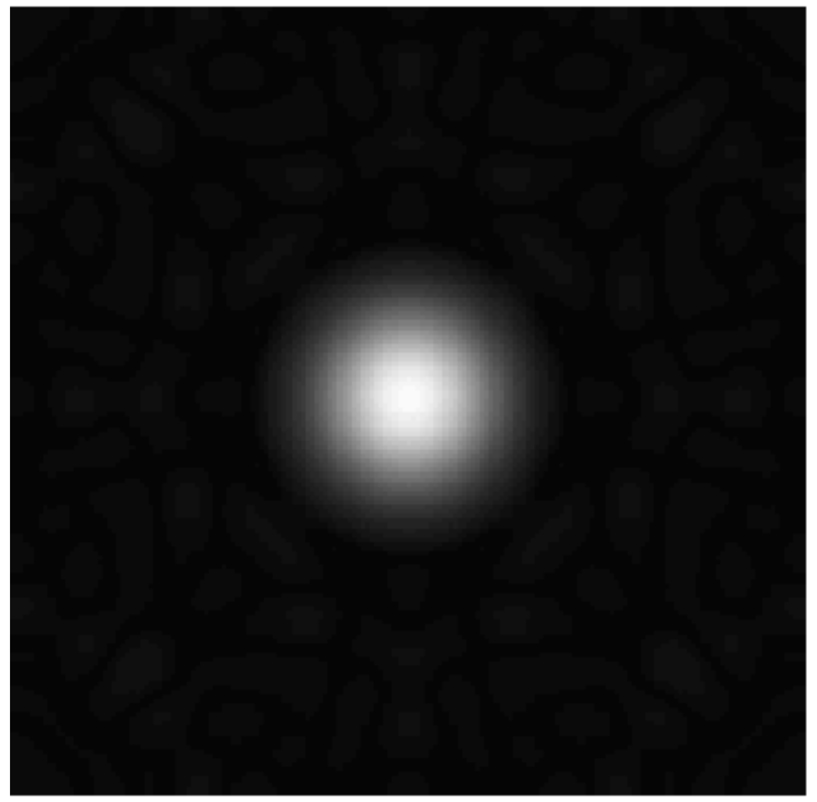
Piazza: What is a low-pass filter?

image



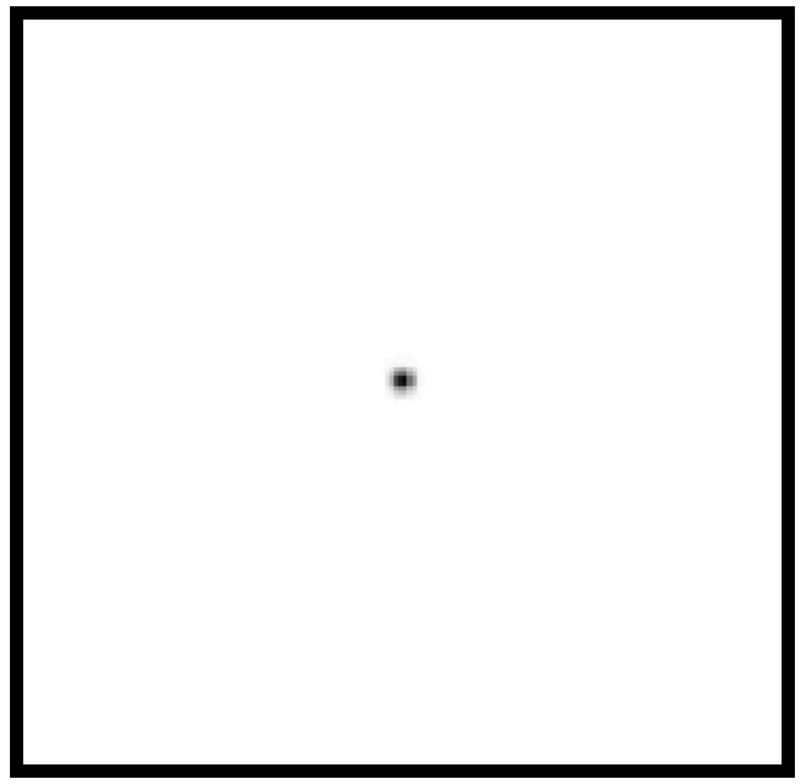
FFT (Mag)

complex
element-wise
multiplication



Low pass

filtered **image**



High pass

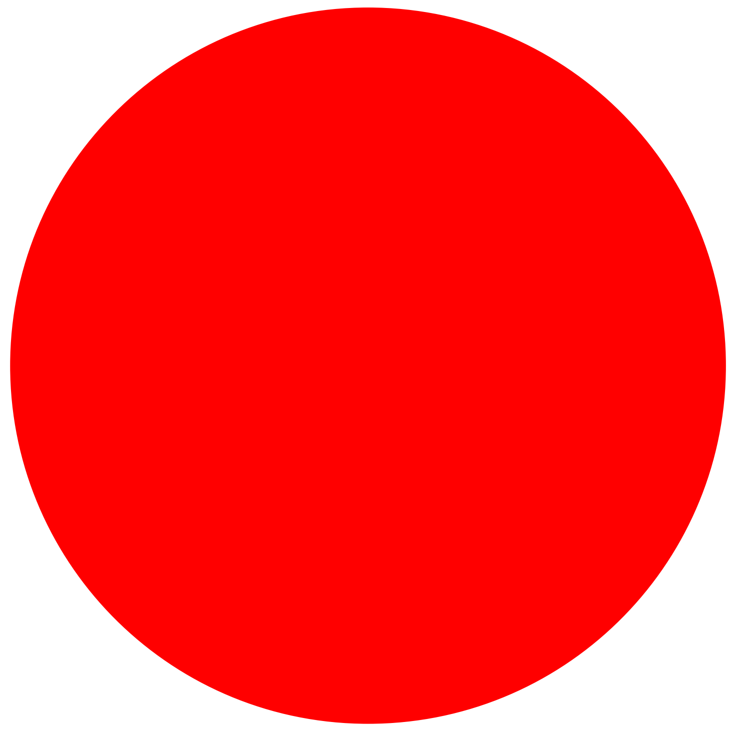
filtered **image**

Low-pass Filtering = “Smoothing”

Box Filter

$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Pillbox Filter



Gaussian Filter

$$\frac{1}{256}\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

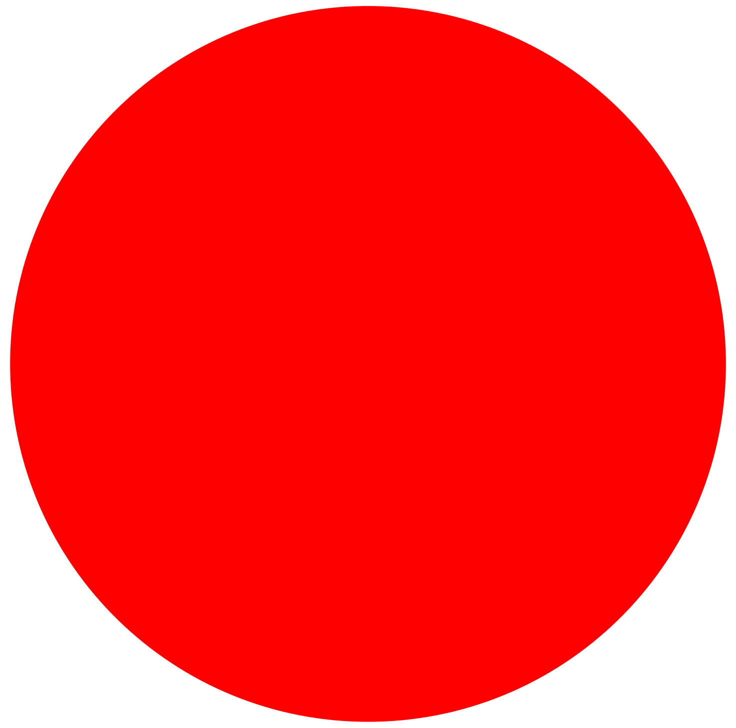
Are all of these **low-pass** filters?

Low-pass Filtering = “Smoothing”

Box Filter

$$\frac{1}{9}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$

Pillbox Filter



Gaussian Filter

$$\frac{1}{256}\begin{bmatrix}1 & 4 & 6 & 4 & 1\\ 4 & 16 & 24 & 16 & 4\\ 6 & 24 & 36 & 24 & 6\\ 4 & 16 & 24 & 16 & 4\\ 1 & 4 & 6 & 4 & 1\end{bmatrix}$$

Are all of these **low-pass** filters?

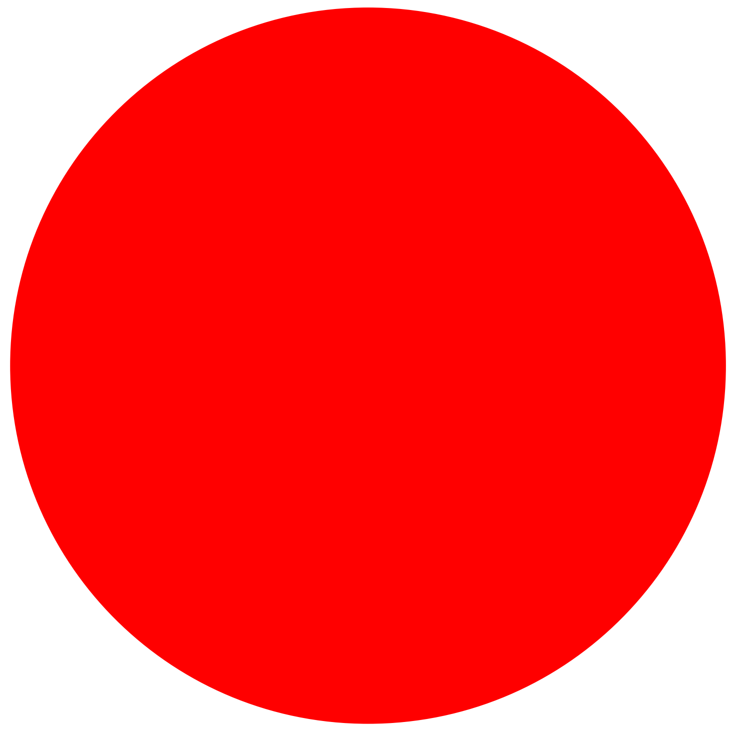
Low-pass filter: Low pass filter filters out all of the high frequency content of the image, only low frequencies remain

Low-pass Filtering = "Smoothing"

Box Filter

$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Pillbox Filter



Gaussian Filter

$$\frac{1}{256}\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

0	0	0	0	0
0	0	0	0	0
0	0	1	0	0
0	0	0	0	0
0	0	0	0	0

Image

Are all of these **low-pass** filters?

Low-pass filter: Low pass filter filters out all of the high frequency content of the image, only low frequencies remain

After long detour …

lets go back to **efficiency**

Speeding Up **Convolution** (The Convolution Theorem)

Convolution **Theorem**:

Let $$i'(x,y) = f(x,y) \otimes i(x,y)$$

then $$\mathcal{I}'(w_x, w_y) = \mathcal{F}(w_x, w_y)\, \mathcal{I}(w_x, w_y)$$

where $\mathcal{I}'(w_x, w_y)$, $\mathcal{F}(w_x, w_y)$, and $\mathcal{I}(w_x, w_y)$ are Fourier transforms of $i'(x,y)$, $f(x,y)$ and $i(x,y)$

At the expense of two **Fourier** transforms and one inverse Fourier transform, convolution can be reduced to (complex) multiplication

Speeding Up **Convolution** (The Convolution Theorem)

General implementation of **convolution**:

At each pixel, (X, Y), there are	$m \times m$	multiplications
There are	$n \times n$	pixels in (X, Y)
Total:	$m^2 \times n^2$	multiplications

Convolution if FFT space:

Cost of FFT/IFFT for image: $\mathcal{O}(n^2 \log n)$

Cost of FFT/IFFT for filter: $\mathcal{O}(m^2 \log m)$

Cost of convolution: $\mathcal{O}(n^2)$

Note: not a function of filter size !!!

Linear Filters: Properties (recall **Lecture 3**)

Let $\otimes$ denote convolution. Let $I(X,Y)$ be a digital image

Superposition: Let F_1 and F_2 be digital filters

$$(F_1 + F_2) \otimes I(X,Y) = F_1 \otimes I(X,Y) + F_2 \otimes I(X,Y)$$

Scaling: Let F be digital filter and let k be a scalar

$$(kF) \otimes I(X,Y) = F \otimes (kI(X,Y)) = k(F \otimes I(X,Y))$$

Shift Invariance: Output is local (i.e., no dependence on absolute position)

An operation is **linear** if it satisfies both **superposition** and **scaling**

Linear Filters: Additional Properties

Let $\otimes$ denote convolution. Let $I(X,Y)$ be a digital image. Let F and G be digital filters

— Convolution is **associative**. That is,

$$G \otimes (F \otimes I(X,Y)) = (G \otimes F) \otimes I(X,Y)$$

— Convolution is **symmetric**. That is,

$$(G \otimes F) \otimes I(X,Y) = (F \otimes G) \otimes I(X,Y)$$

Convolving $I(X,Y)$ with filter F and then convolving the result with filter G can be achieved in single step, namely convolving $I(X,Y)$ with filter $G \otimes F = F \otimes G$

Note: Correlation, in general, is **not associative**.