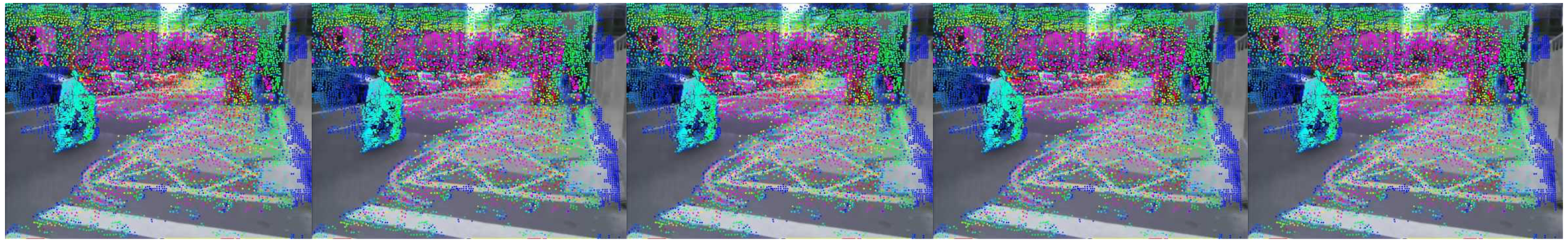
THE UNIVERSITY OF BRITISH COLUMBIA

CPSC 425: Computer Vision



Lecture 17: Optical Flow (cont.)

Menu for Today (**March 11, 2020**)

Topics:

— Optical Flow (cont)
— Classification
— Naive Bayes Classifier
— Bayes' Risk

Redings:

— **Today's** Lecture: Forsyth & Ponce (2nd ed.) 15.1, 15.2
— **Next** Lecture: Forsyth & Ponce (2nd ed.) 16.1.3, 16.1.4, 16.1.9

Reminders:

— **Assignment 4**: Local Invariant Features and RANSAC due **Tuesday**

— **Midterm** graded. Grades will be released soon.

Today's "**fun**" Example: Visual Microphone

Follow-up work to previous lecture's example of **Eulerian** video magnification

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Lecture 16: Re-cap

Optical flow is the apparent motion of brightness patterns in the image

Applications

— image and video stabilization in digital cameras, camcorders

— motion-compensated video compression schemes such as MPEG

— image registration for medical imaging, remote sensing

— action recognition

— motion segmentation

Lecture 16: Re-cap

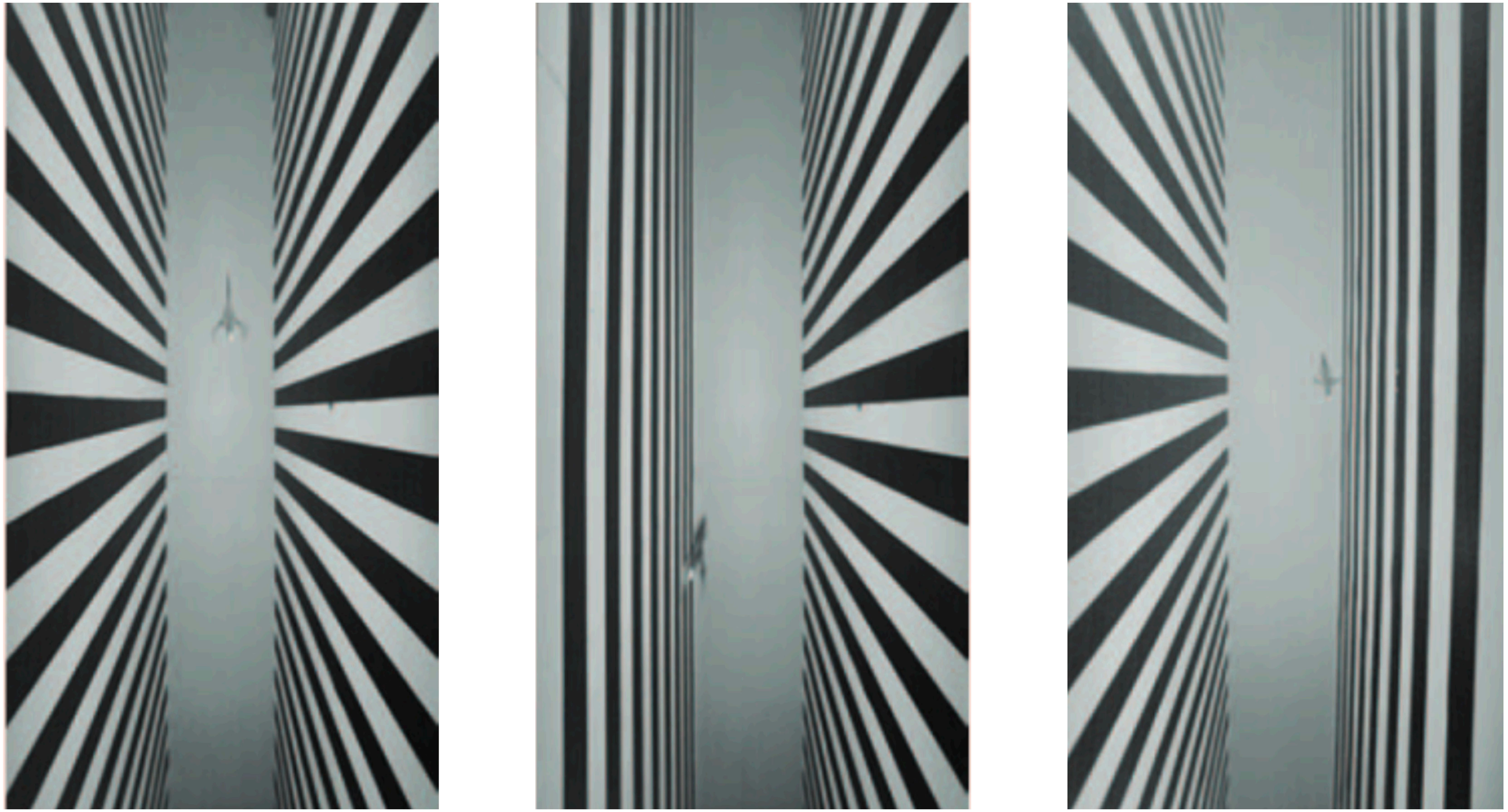
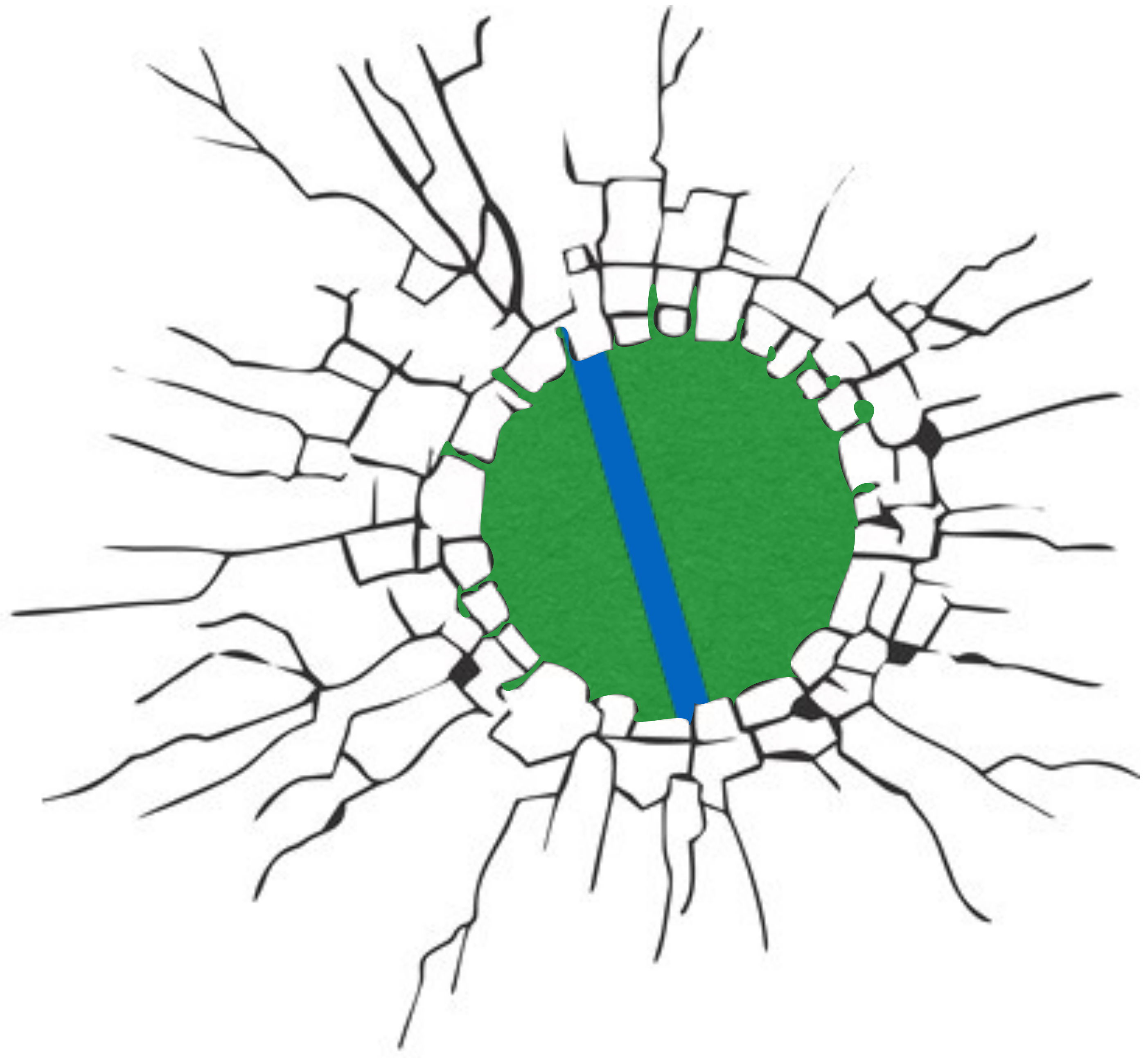


Figure credit: M. Srinivasan

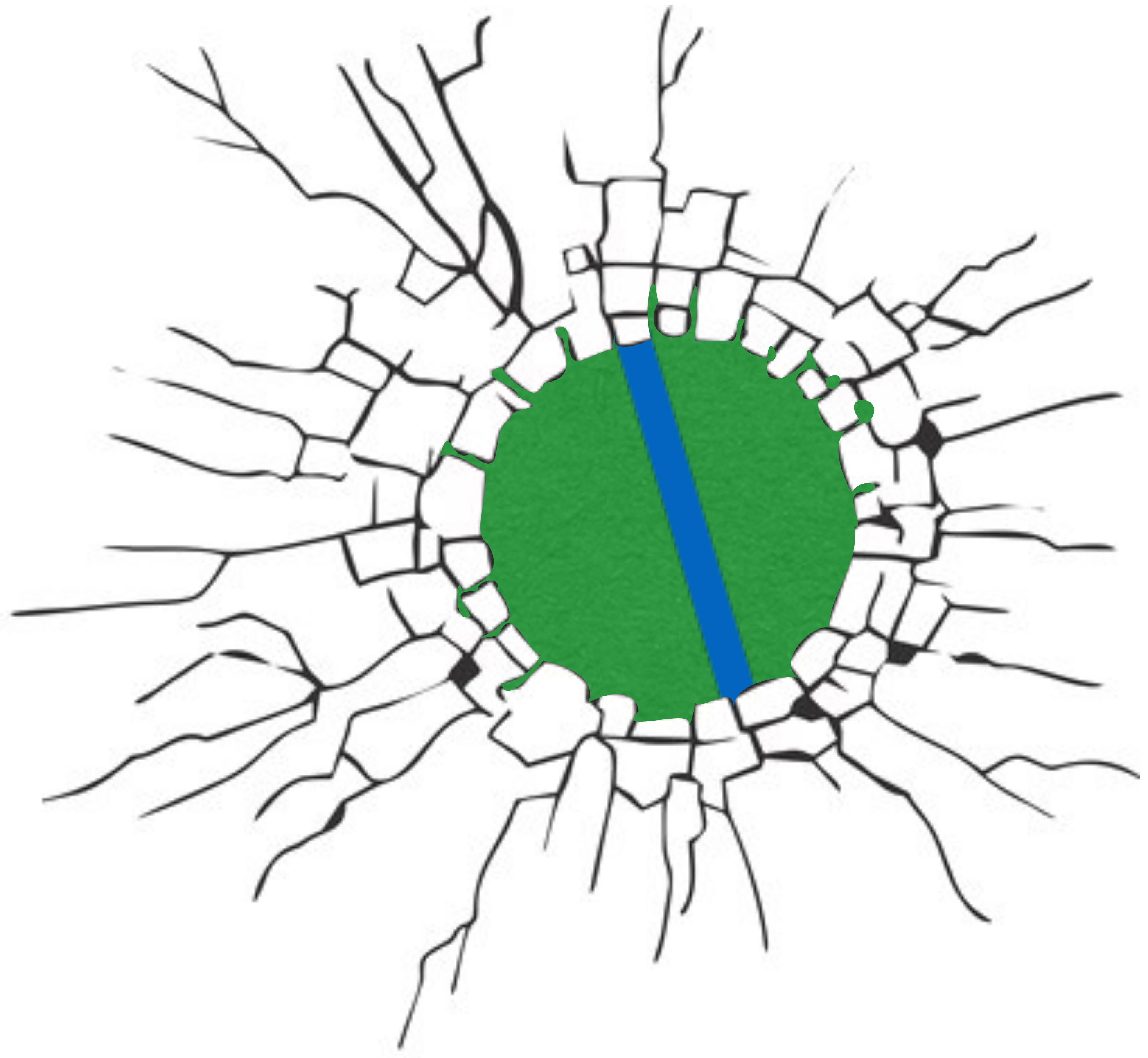
Aperture Problem



In which direction is the line moving?

Image Credit: Ioannis (Yannis) Gkioulekas (CMU)

Aperture Problem



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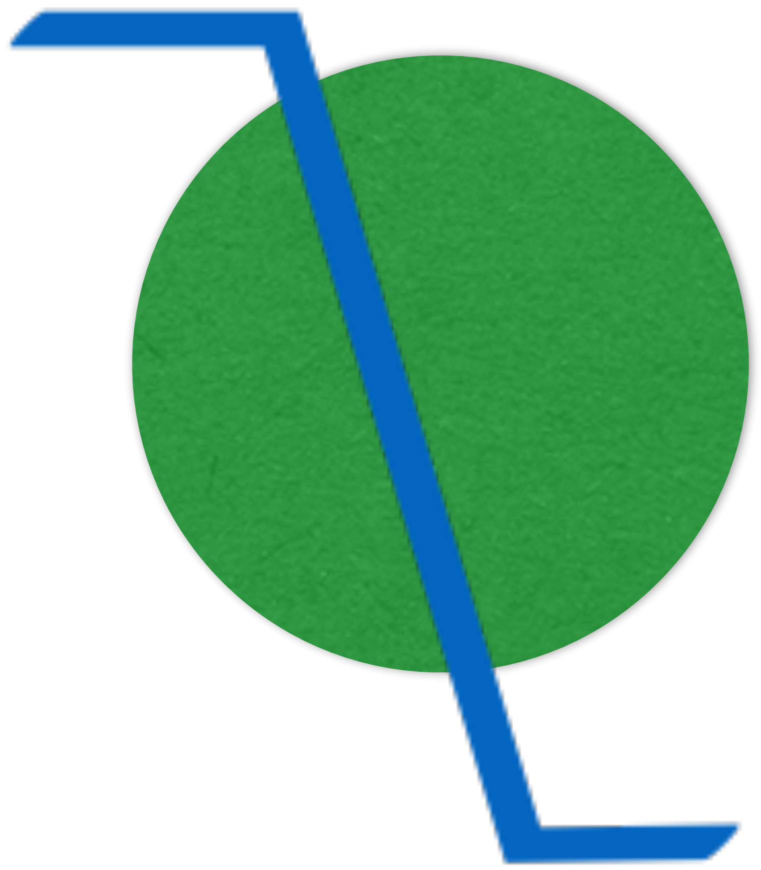

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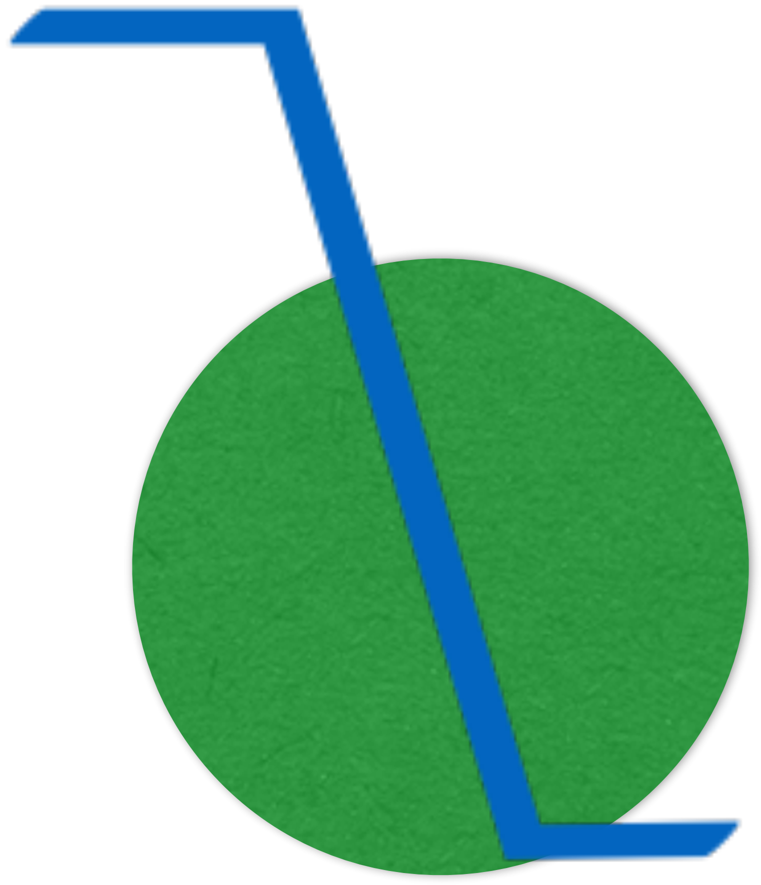

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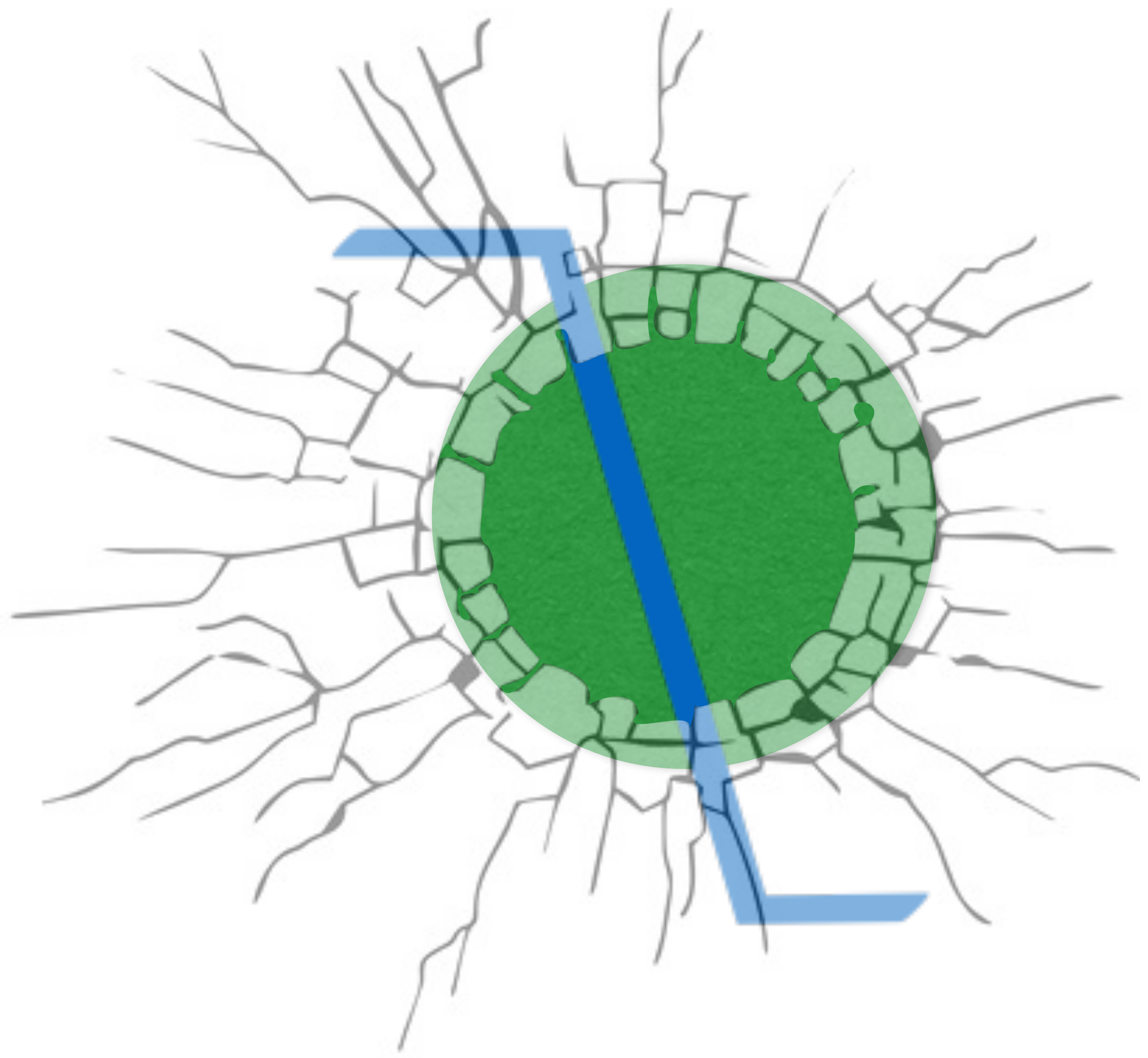

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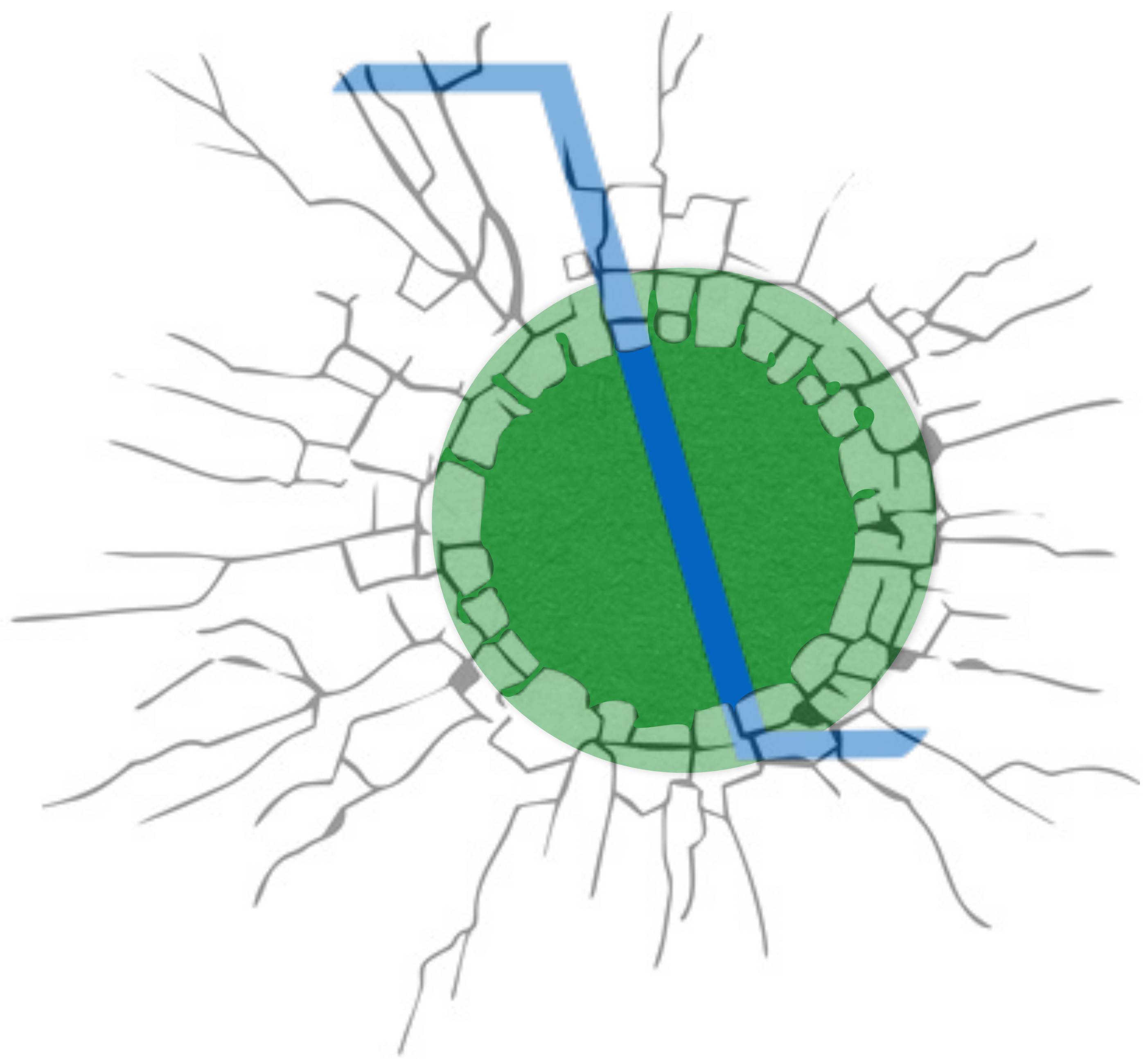

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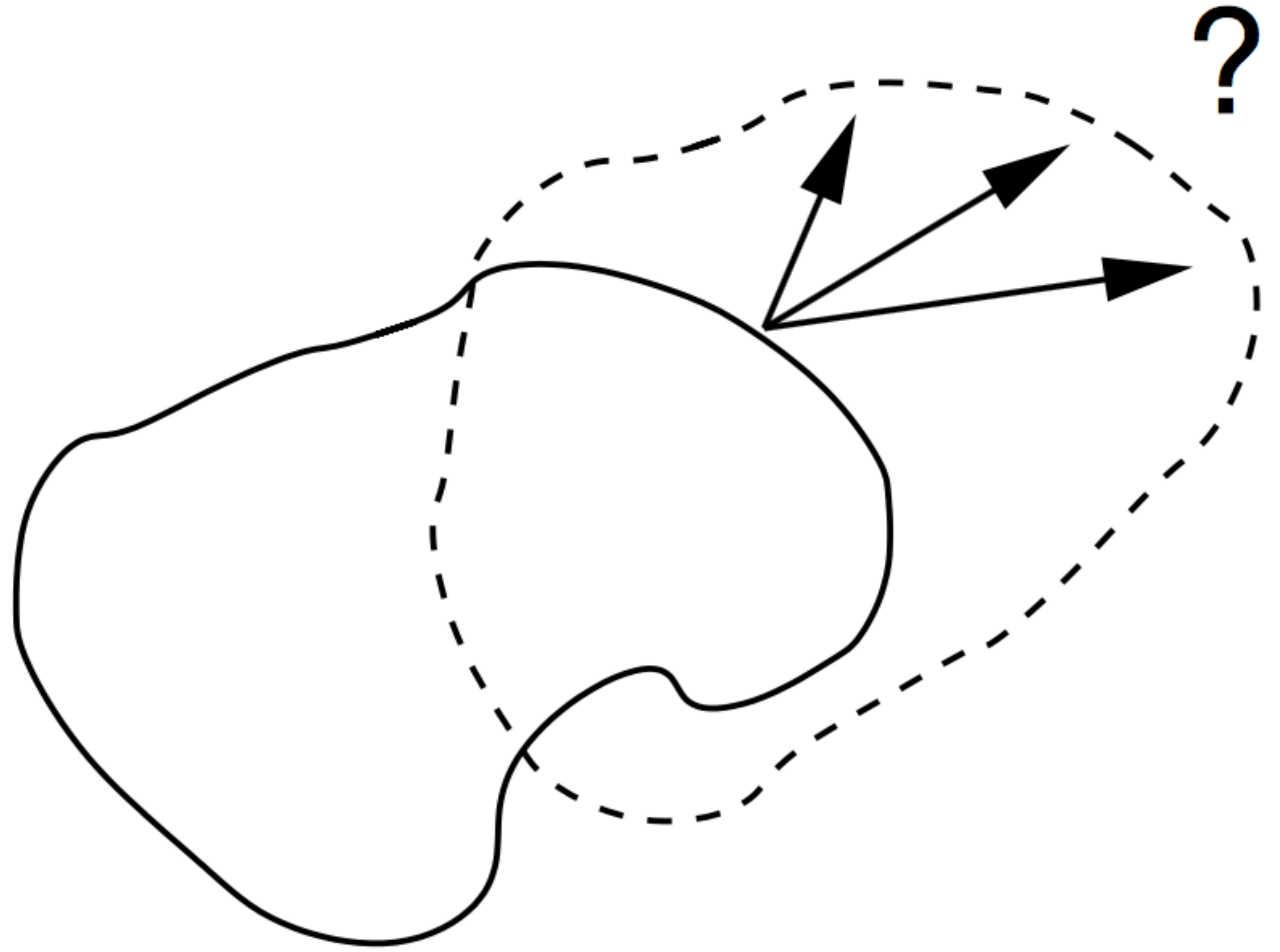
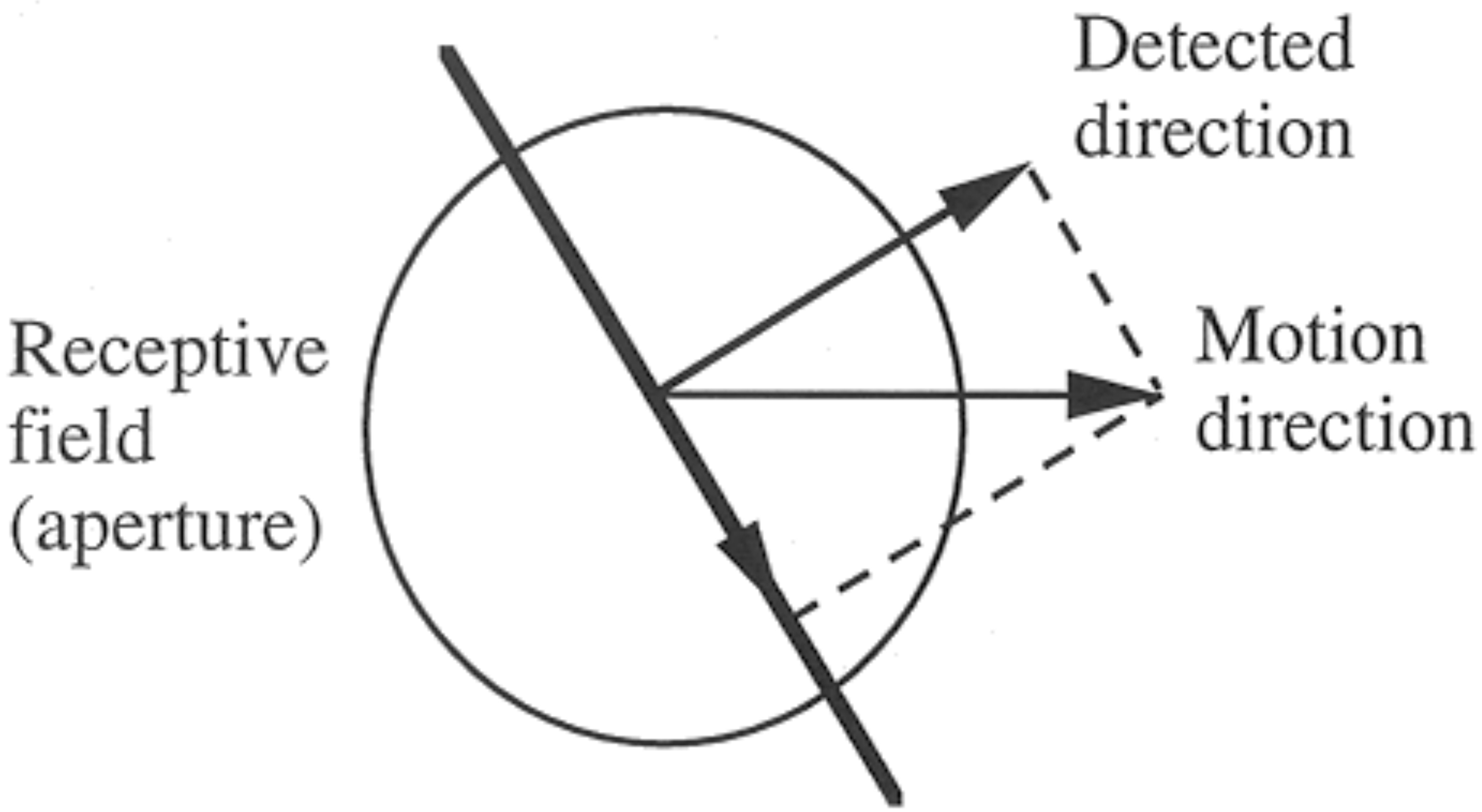


— Without distinct features to track, the true visual motion is ambiguous

— Locally, one can compute only the component of the visual motion in the direction perpendicular to the contour

Aperture Problem



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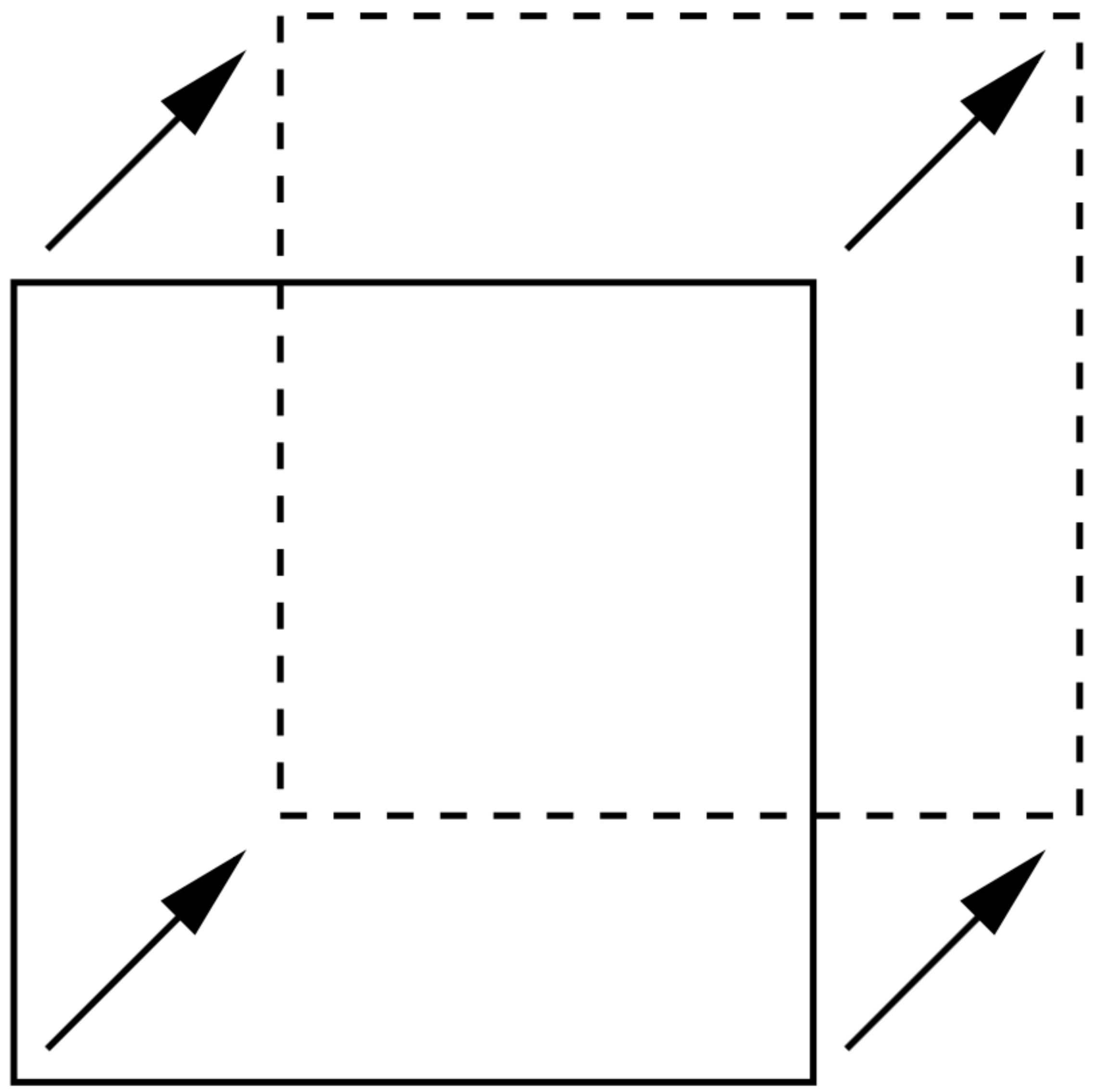
Visual **Motion**



Visual motion is determined when there are distinct features to track, provided:

— the features can be detected and localized accurately; and

— the features can be correctly matched over time

Motion as **Matching**

Representation	Result is. . .
Point/feature based	(very) sparse
Contour based	(relatively) sparse
(Differential) gradient based	dense

Optical Flow **Constraint Equation**

Consider image intensity also to be a function of time, t. We write

$$I(x, y, t)$$

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Applying the **chain rule for differentiation**, we obtain

$$\frac{dI(x,y,t)}{dt} = I_x \frac{dx}{dt} + I_y \frac{dy}{dt} + I_t$$

where subscripts denote partial differentiation

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Define $u = \dfrac{dx}{dt}$ and $v = \dfrac{dy}{dt}$. Then $[u, v]$ is the 2-D motion and the space of all such u and v is the **2-D velocity space**

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Suppose $\frac{dI(x,y,t)}{dt} = 0$. Then we obtain the (classic) **optical flow constraint equation**

$$I_x u + I_y v + I_t = 0$$

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Optical Flow **Constraint Equation**

What does this mean, and why is it reasonable?

Suppose $\dfrac{dI(x, y, t)}{dt} = 0$. Then we obtain the (classic) **optical flow constraint equation**

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Optical Flow **Constraint Equation**

Scene point moving through image sequence

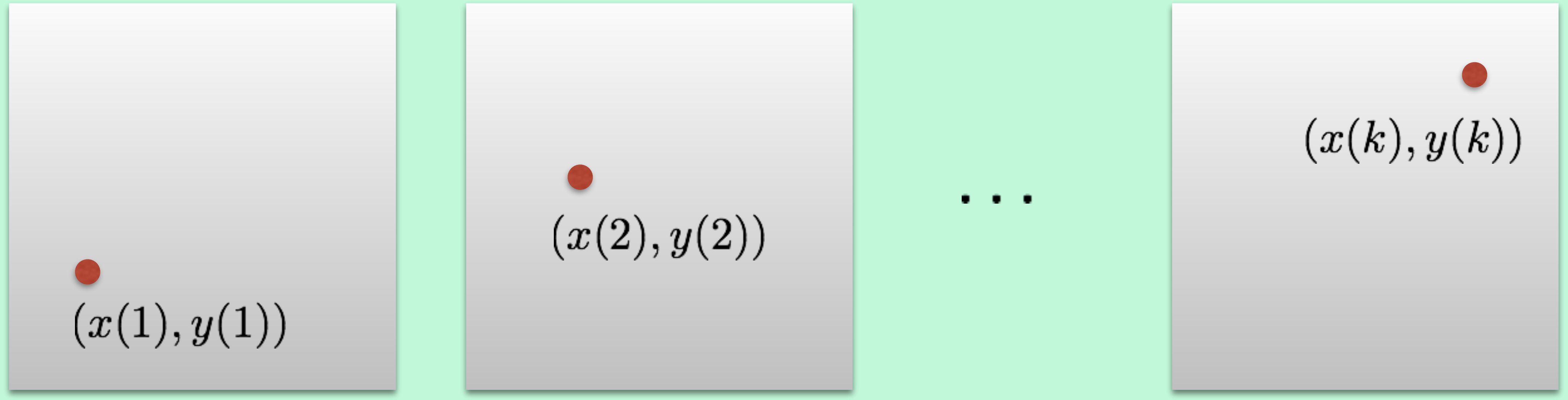

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Image Credit: Ioannis (Yannis) Gkioulekas (CMU)

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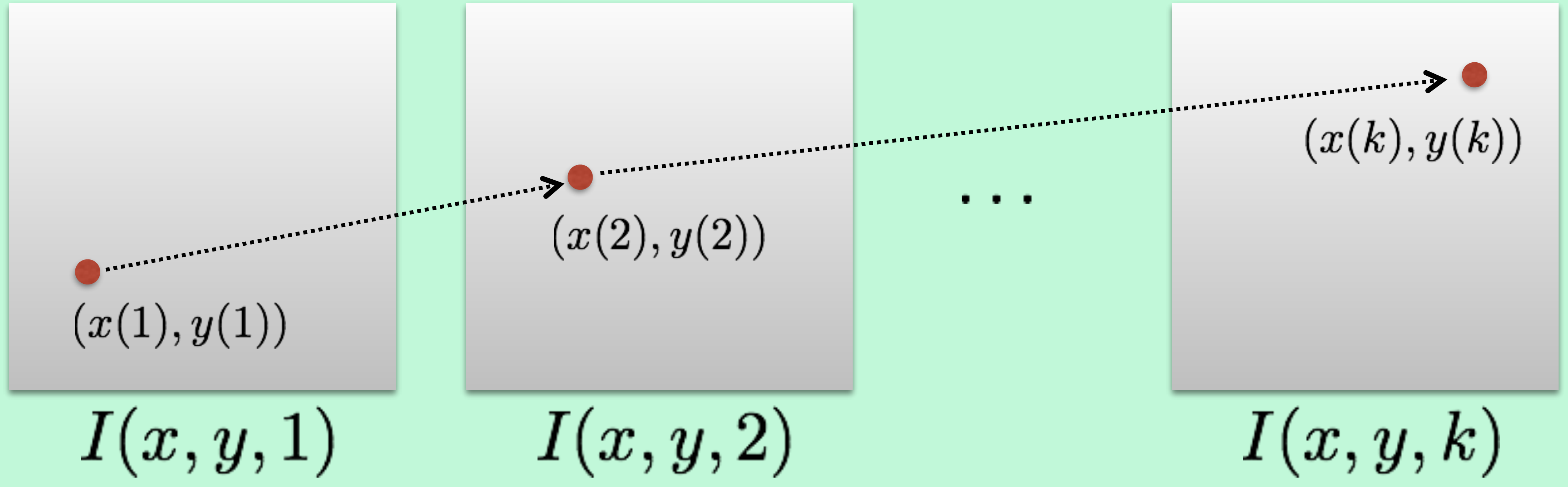

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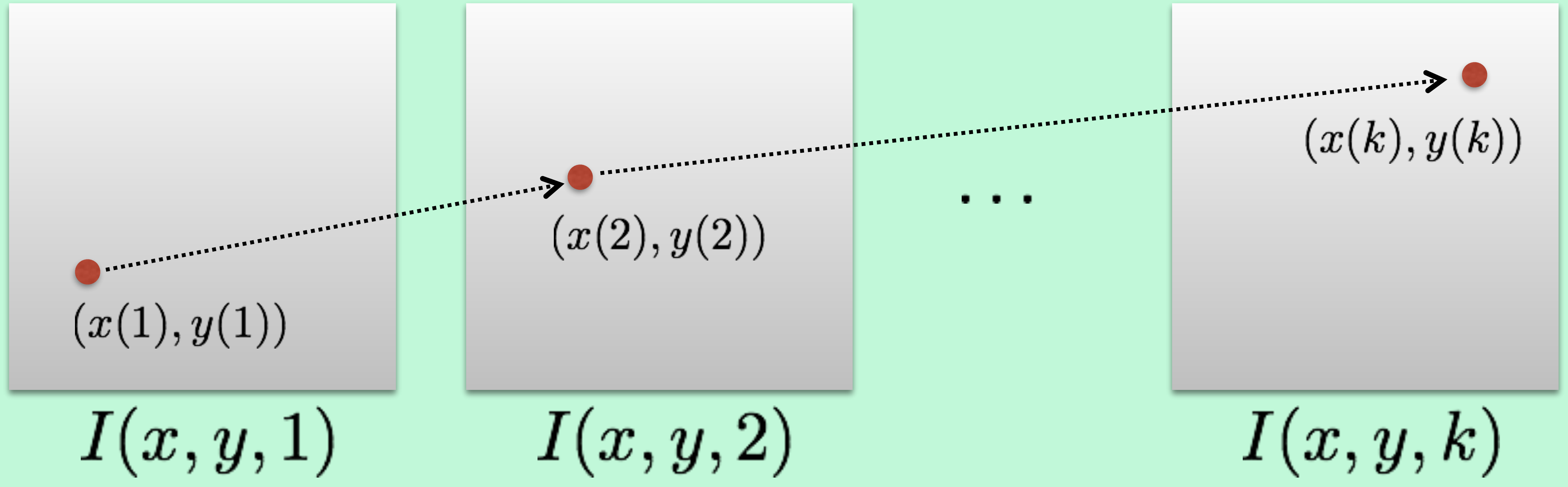
Optical Flow **Constraint Equation**

Brightness Constancy Assumption: Brightness of the point remains the same



$$I(x(t), y(t), t) = C$$

constant

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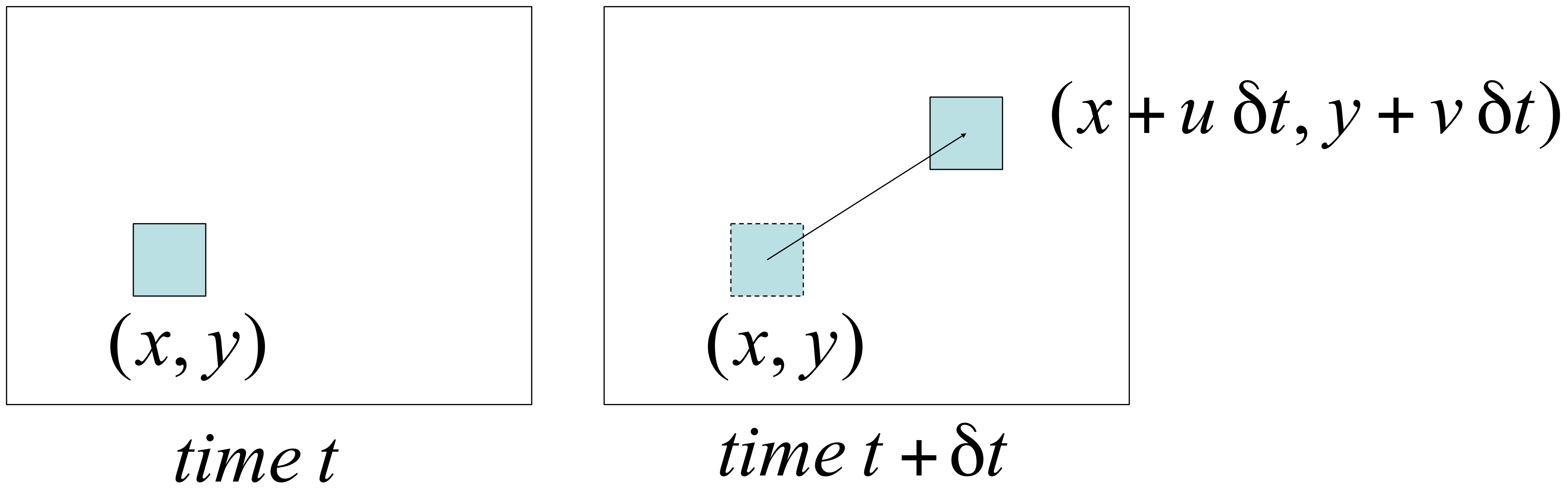
Image Credit: Ioannis (Yannis) Gkioulekas (CMU)

Aside: Derivation of Optical Flow Constraint

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

For small space-time step, brightness of a point is the same



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For small space-time step, brightness of a point is the same

Insight:
If the time step is really small,
we can *linearize* the intensity function

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

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$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

Multivariable Taylor Series Expansion

(First order approximation, two variables)

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) - f_y(a,b)(y-b)$$

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$$I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = I(x, y, t)$$ assuming small motion

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Multivariable Taylor Series Expansion

(First order approximation, two variables)

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) - f_y(a,b)(y-b)$$

partial derivative

$$I(x,y,t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = I(x,y,t)$$

fixed point

assuming small motion

cancel terms

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assuming small motion

$$\frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = 0$$

divide by δt

take limit $\delta t \rightarrow 0$

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take limit $\delta t \to 0$

$$\boxed{\frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0}$$ **Brightness Constancy Equation**

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

How do we **compute** …

$$I_x u + I_y v + I_t = 0$$

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How do we **compute** …

$$I_x u + I_y v + I_t = 0$$

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

spatial derivative

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spatial derivative

Forward difference
Sobel filter
Scharr filter
…

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$$I_t = \frac{\partial I}{\partial t}$$

temporal derivative

Forward difference
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Scharr filter
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$$I_t = \frac{\partial I}{\partial t}$$

temporal derivative

Frame differencing

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

Frame Differencing: Example

$$t+1$$

1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	10	10	10
1	1	10	10	10
1	1	10	10	10

-

$$t$$

1	1	1	1	1
1	1	1	1	1
1	10	10	10	10
1	10	10	10	10
1	10	10	10	10
1	10	10	10	10

=

$$I_t = \frac{\partial I}{\partial t}$$

0	0	0	0	0
0	0	0	0	0
0	-9	-9	-9	-9
0	-9	0	0	0
0	-9	0	0	0
0	-9	0	0	0

(example of a forward temporal difference)

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

t

1	1	1	1	1
1	1	1	1	1
1	10	10	10	10
1	10	10	10	10
1	10	10	10	10
1	10	10	10	10

$t+1$

1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	10	10	10
1	1	10	10	10
1	1	10	10	10

$$I_x = \frac{\partial I}{\partial x}$$

-	0	0	0	-
-	0	0	0	-
-	9	0	0	-
-	9	0	0	-
-	9	0	0	-
-	9	0	0	-

-1 0 1

$$I_y = \frac{\partial I}{\partial y}$$

-	-	-	-	-
0	0	0	0	0
0	9	9	9	9
0	0	0	0	0
0	0	0	0	0
-	-	-	-	-

-1
0
1

$$I_t = \frac{\partial I}{\partial t}$$

0	0	0	0	0
0	0	0	0	0
0	-9	-9	-9	-9
0	-9	0	0	0
0	-9	0	0	0
0	-9	0	0	0

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

How do we **compute** …

$$I_x u + I_y v + I_t = 0$$

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

spatial derivative

$$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$$

optical flow

$$I_t = \frac{\partial I}{\partial t}$$

temporal derivative

Forward difference
Sobel filter
Scharr filter
…

How do you compute this?

Frame differencing

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

How do we **compute** …

$$I_x u + I_y v + I_t = 0$$

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

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optical flow

$$I_t = \frac{\partial I}{\partial t}$$

temporal derivative

Forward difference
Sobel filter
Scharr filter
…

We need to solve for this!
(this is the unknown in the optical flow problem)

Frame differencing

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

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optical flow

$$I_t = \frac{\partial I}{\partial t}$$

temporal derivative

Forward difference
Sobel filter
Scharr filter
…

(u, v)

Solution lies on a line

Cannot be found uniquely with a single constraint

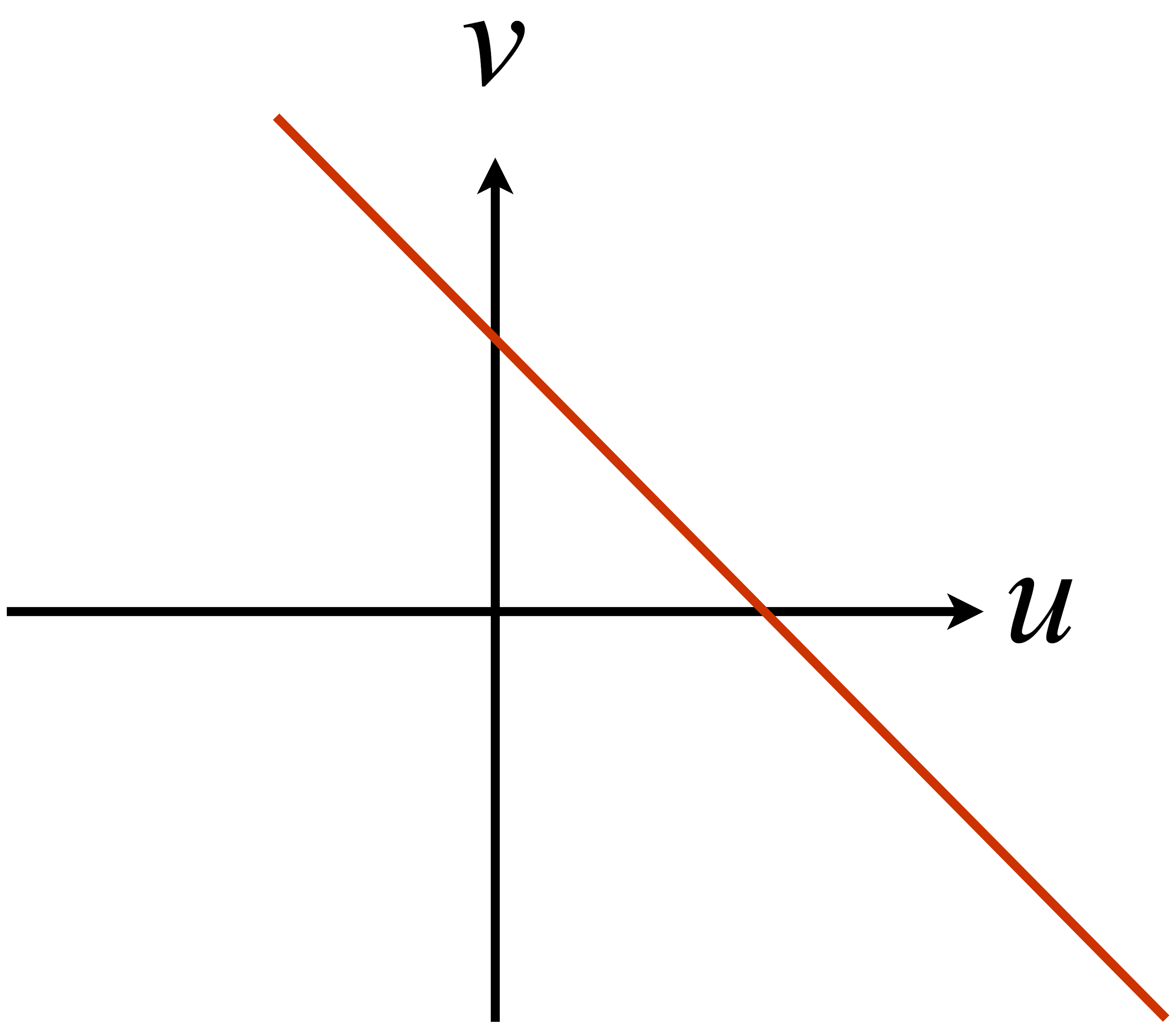
Frame differencing

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

Optical Flow **Constraint Equation**

$$\boxed{I_x u + I_y v + I_t = 0}$$

many combinations of u and v will satisfy the equality



Equation determines a **straight line** in velocity space

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

Lucas-Kanade

Observations:

1. The 2-D motion, $[u, v]$, at a given point, $[x, y]$, has two degrees-of-freedom

2. The partial derivatives, I_x, I_y, I_t, provide one constraint

3. The 2-D motion, $[u, v]$, cannot be determined locally from I_x, I_y, I_t alone

Lucas-Kanade

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Lucas–Kanade Idea:

Obtain additional local constraint by computing the partial derivatives, I_x, I_y, I_t, in a window centered at the given $[x, y]$

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Lucas–Kanade Idea:

Obtain additional local constraint by computing the partial derivatives, I_x, I_y, I_t, in a window centered at the given $[x, y]$

Constant Flow Assumption: nearby pixels will likely have same optical flow

Lucas-Kanade

Optical Flow Constraint Equation: $I_x u + I_y v + I_t = 0$

Suppose $[x_1, y_1] = [x, y]$ is the (original) center point in the window. Let $[x_2, y_2]$ be any other point in the window. This gives us two equations that we can write

$$I_{x_1} u + I_{y_1} v = -I_{t_1}$$
$$I_{x_2} u + I_{y_2} v = -I_{t_2}$$

and that can be solved locally for u and v as

$$\begin{bmatrix} u \\ v \end{bmatrix} = -\begin{bmatrix} I_{x_1} & I_{y_1} \\ I_{x_2} & I_{y_2} \end{bmatrix}^{-1} \begin{bmatrix} I_{t_1} \\ I_{t_2} \end{bmatrix}$$

provided that u and v are the same in both equations and provided that the required matrix inverse exists.

Lucas-Kanade

Optical Flow Constraint Equation: $I_x u + I_y v + I_t = 0$

Considering all n points in the window, one obtains

$$\begin{aligned} I_{x_1} u + I_{y_1} v &= -I_{t_1} \\ I_{x_2} u + I_{y_2} v &= -I_{t_2} \\ &\vdots \\ I_{x_n} u + I_{y_n} v &= -I_{t_n} \end{aligned}$$

which can be written as the matrix equation

$$\mathbf{A}\mathbf{v} = \mathbf{b}$$

where $\mathbf{v} = [u, v]^T$, $\mathbf{A} = \begin{bmatrix} I_{x_1} & I_{y_1} \\ I_{x_2} & I_{y_2} \\ \vdots & \vdots \\ I_{x_n} & I_{y_n} \end{bmatrix}$ and $\mathbf{b} = -\begin{bmatrix} I_{t_1} \\ I_{t_2} \\ \vdots \\ I_{t_n} \end{bmatrix}$

Lucas-Kanade

The standard least squares solution, $\bar{\mathbf{v}}$, to is

$$\bar{\mathbf{v}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$$

again provided that u and v are the same in all equations and provided that the rank of $\mathbf{A}^T\mathbf{A}$ is 2 (so that the required inverse exists)

Lucas-Kanade

The standard least squares solution, $\bar{\mathbf{v}}$, to is

$$\begin{bmatrix} I_{x_1} & I_{x_2} & \cdots & I_{x_n} \\ I_{y_1} & I_{y_2} & \cdots & I_{y_n} \end{bmatrix} \quad \begin{bmatrix} I_{x_1} & I_{y_1} \\ I_{x_2} & I_{y_2} \\ \vdots & \vdots \\ I_{x_n} & I_{y_n} \end{bmatrix}$$

$$\bar{\mathbf{v}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$$

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Lucas-Kanade

Note that we can explicitly write down an expression for $\mathbf{A}^T\mathbf{A}$ as

$$\mathbf{A}^T\mathbf{A} = \left[\begin{array}{cc} \sum I_x^2 & \sum I_x I_y \\ \sum I_x I_y & I_y^2 \end{array} \right]$$

which is identical to the matrix $\mathbf{C}$ that we saw in the context of Harris corner detection

Lucas-Kanade

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What does that mean?

Lucas-Kanade **Summary**

A dense method to compute motion, $[u, v]$ at every location in an image

Key Assumptions:

1. Motion is slow enough and smooth enough that differential methods apply (i.e., that the partial derivatives, I_x, I_y, I_t, are well-defined)

2. The optical flow constraint equation holds (i.e., $\frac{dI(x,y,t)}{dt} = 0$)

3. A window size is chosen so that motion, $[u, v]$, is constant in the window

4. A window size is chosen so that the rank of $\mathbf{A}^T\mathbf{A}$ is 2 for the window

Aside: Optical Flow Smoothness Constraint

Many methods trade off a ‘departure from the optical flow constraint’ cost with a ‘departure from smoothness’ cost.

The optimization objective to minimize becomes

$$E = \int\int (I_x u + I_y v + I_t)^2 + \lambda(|| \bigtriangledown u||^2 + || \bigtriangledown v||^2)$$

where λ is a weighing parameter.

Horn-Schunck Optical Flow

$$\min_{\boldsymbol{u},\boldsymbol{v}} \sum_{i,j} \Big\{ \overbrace{E_s(i,j)}^{\text{smoothness}} + \underbrace{\lambda}_{\text{weight}} \overbrace{E_d(i,j)}^{\text{brightness constancy}} \Big\}$$

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

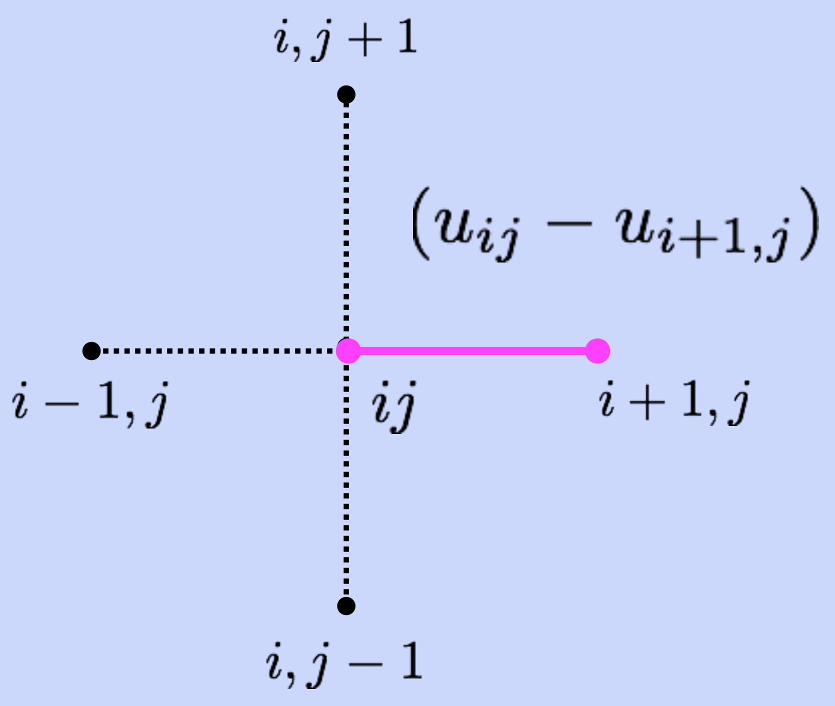
Horn-Schunck Optical Flow

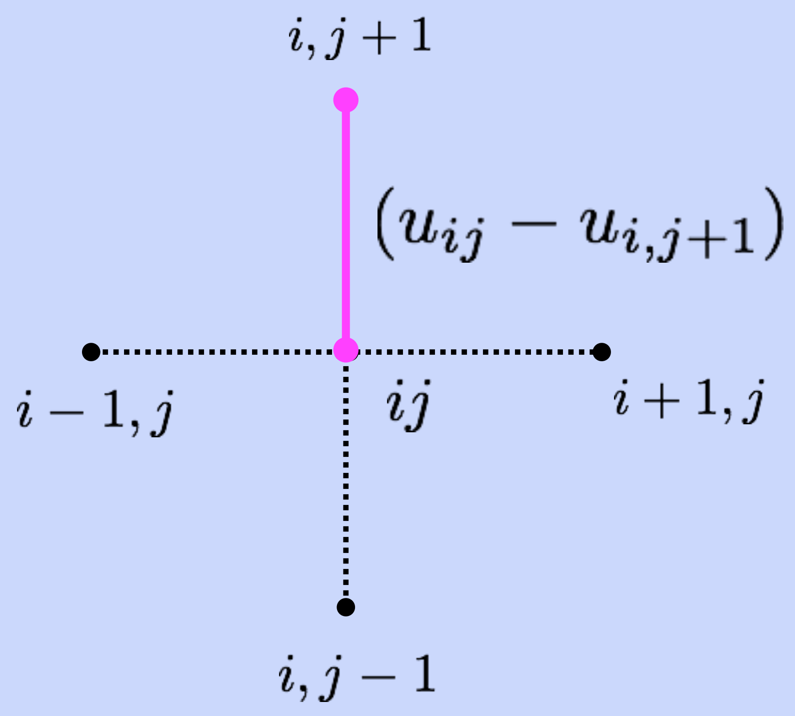
Brightness constancy

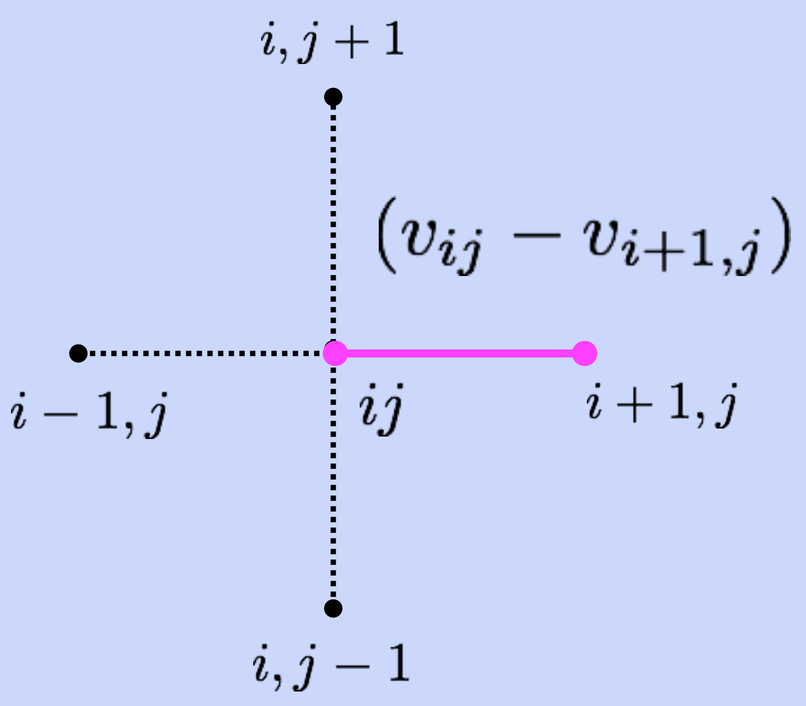
$$E_d(i,j) = \left[I_x u_{ij} + I_y v_{ij} + I_t \right]^2$$

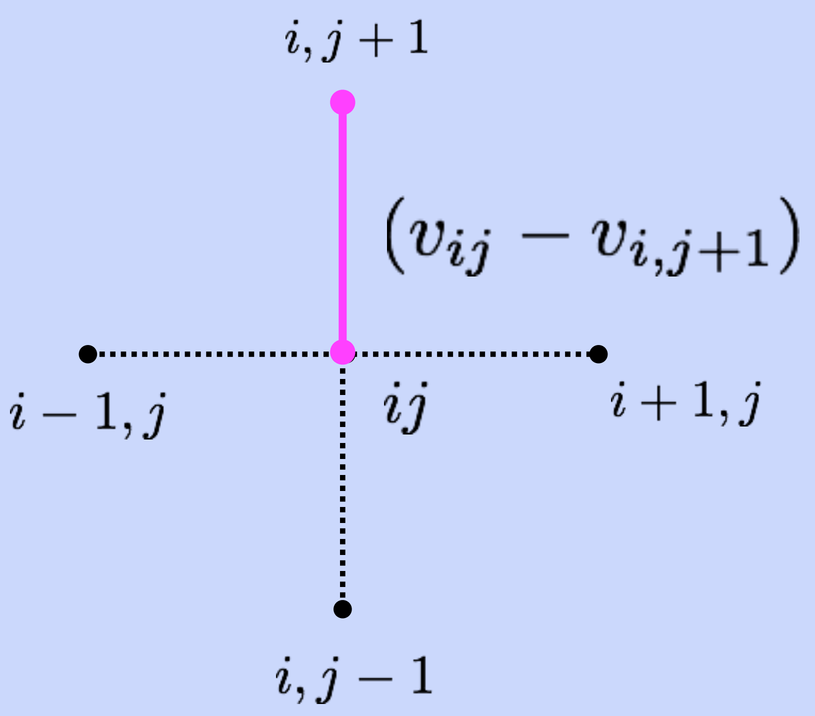
Smoothness

$$E_s(i,j) = \frac{1}{4}\left[(u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \right]$$









Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)

Summary

Motion, like binocular stereo, can be formulated as a matching problem. That is, given a scene point located at (x_0, y_0) in an image acquired at time t_0, what is its position, (x_1, y_1), in an image acquired at time t_1?

Assuming image intensity does not change as a consequence of motion, we obtain the (classic) **optical flow constraint equation**

$$I_x u + I_y v + I_t = 0$$

where $[u, v]$, is the 2-D motion at a given point, $[x, y]$, and I_x, I_y, I_t are the partial derivatives of intensity with respect to x, y, and t

Lucas–Kanade is a dense method to compute the motion, $[u, v]$, at every location in an image