CPSC 425: Computer Vision

Image Credit: https://en.wikibooks.org/wiki/Analog_and_Digital_Conversion/Nyquist_Sampling_Rate

Lecture 5: Sampling

( unless otherwise stated slides are taken or adopted from Bob Woodham, Jim Little and Fred Tung )
Menu for Today (January 17, 2018)

Topics:

— Sampling
— Aliasing

— Bandlimited Signals
— Nyquist rate

Readings:

— Today’s Lecture: Forsyth & Ponce (2nd ed.) 4.4
— Next Lecture: Forsyth & Ponce (2nd ed.) 4.5

Reminders:

— Assignment 1: Image Filtering and Hybrid Images due January 25th
Problem: How do we go from the optics of image formation to digital images as arrays of numbers?

Key Idea(s): Sampling and the notion of band limited functions

Theory: Sampling Theory
Images are a *discrete*, or *sampled*, representation of a continuous world.
What is an Image?

Up to now provided a **physical characterization**
— image formation as a problem in physics/optics

Now provide a **mathematical characterization**
— to understand how to represent images digitally
— to understand how to compute with images
Continuous Case

“Image” suggests a 2D surface whose appearance varies from point–to–point — the surface typically is a plane (but might be curved, e.g., as is with an eye)

Appearance can be **Grayscale** (Black and White) or **Colour**

In **Grayscale**, variation in appearance can be described by a single parameter corresponding to the amount of light reaching the image at a given point in a given time
Continuous Case

Denote the image as a function, $i(x, y)$, where $x$ and $y$ are spatial variables.

Aside: The convention for this section is to use lower case letters for the continuous case and upper case letters for the discrete case.
Continuous Case: Observations

– \( i(x, y) \) is a \textbf{real-valued function} of \textbf{real spatial variables}, \( x \) and \( y \)
Recall: Pinhole Camera

Forsyth & Ponce (2nd ed.) Figure 1.2
Continuous Case: Observations

— $i(x, y)$ is a **real-valued function** of real **spatial variables**, $x$ and $y$

— $i(x, y)$ is **bounded above and below**. That is

\[
0 \leq i(x, y) \leq M
\]

for some maximum brightness $M$
Continuous Case: Observations

- $i(x, y)$ is a **real-valued function** of **real spatial variables**, $x$ and $y$

- $i(x, y)$ is **bounded above and below**. That is

  $$0 \leq i(x, y) \leq M$$

  for some maximum brightness $M$

- $i(x, y)$ is **bounded in extent**. That is, $i(x, y)$ is non-zero (i.e., strictly positive) over, at most, a bounded region
Continuous Case

— Images also can be considered a function of time. Then, we write \( i(x, y, t) \) where \( x \) and \( y \) are spatial variables and \( t \) is a **temporal variable**.

— To make the dependence of brightness on wavelength explicit, we can instead write \( i(x, y, t, \lambda) \) where \( x, y \) and \( t \) are as above and where \( \lambda \) is a **spectral variable**.

— More commonly, we think of “color” already as discrete and write

\[
i_R(x, y) \\
i_G(x, y) \\
i_B(x, y)
\]

for specific colour channels, R, G and B.
**Discrete Case**

**Idea**: Superimpose (regular) grid on continuous image

Sample the underlying continuous image according to the **tessellation** imposed by the grid
Discrete Case

The diagram illustrates a grid with a pixel labeled as \(i(x,y)\). The grid is defined by the x and y axes, with the pixel located at a specific point within the grid. The text "pixel" is mentioned in the diagram, indicating the representation of a single point in the grid.
Each grid cell is called a picture element (pixel).

Denote the discrete image as $I(X, Y)$.

We can store the pixels in a matrix or array.
**Discrete Case**

**Question:** How to sample?
- Sample brightness at the point?
- “Average” brightness over entire pixel?

**Answer:**
- Point sampling is useful for theoretical development
- Area-based sampling occurs in practice
Discrete Case

**Question:** What about the brightness samples themselves?
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**Answer:** We make values of $I(X, Y)$ discrete as well.

Recall: $0 \leq i(x, y) \leq M$

We divide the range $[0, M]$ into a finite number of equivalence classes. This is called **quantization**.

The values are called **grey-levels**.
Quantization is a topic in its own right for now, a simple linear scheme is sufficient.

Suppose \( n \) bits-per-pixel are available. One can divide the range \([0, M]\) into evenly spaced intervals as follows:

\[
\left\lfloor \frac{i(x, y)}{M} (2^n - 1) + 0.5 \right\rfloor
\]

where \( \lfloor \cdot \rfloor \) is floor (i.e., greatest integer less than or equal to).

Typically \( n = 8 \) resulting in grey-levels in the range \([0, 255]\)
It is clear that *some* information may be lost when we work on a discrete pixel grid.

Forsyth & Ponce (2nd ed.) Figure 4.7
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**Sampling Theory (informal)**

**Question:** When is $I(X, Y)$ an exact characterization of $i(x, y)$?
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**Question (modified)**: When can we reconstruct $i(x, y)$ exactly from $I(X, Y)$?
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**Intuition**: Reconstruction involves some kind of interpolation.
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**Question**: When is $I(X, Y)$ an exact characterization of $i(x, y)$?

**Question (modified)**: When can we reconstruct $i(x, y)$ exactly from $I(X, Y)$?

**Intuition**: Reconstruction involves some kind of **interpolation**

**Heuristic**: When in doubt, consider simple cases
**Sampling Theory (informal)**

**Case 0**: Suppose $i(x, y) = k$ (with $k$ being one of our gray levels)

**Note**: we use equidistant sampling at integer values for convenience, in general, sampling doesn’t need to be equidistant
Case 0: Suppose $i(x, y) = k$ (with $k$ being one of our gray levels)

This is easy!
Case 0: Suppose \( i(x, y) = k \) (with \( k \) being one of our gray levels)

\[
I(X, Y) = k. \text{ Any standard interpolation function would give } i(x, y) = k \text{ for non-integer } x \text{ and } y \text{ (irrespective oh how coarse the sampling is)}
\]
Case 0: Suppose $i(x, y)$ has a discontinuity not falling precisely at integer $x, y$. 

\[ i(x) \]

\[ k_0 \]

\[ k_1 \]

\[ x \]
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We cannot reconstruct $i(x, y)$ exactly because we can never know exactly where the discontinuity lies.
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We cannot reconstruct $i(x, y)$ exactly because we can never know exactly where the discontinuity lies.

This is **impossible**!
Sampling Theory (informal)

**Question:** How do we close the gap between “easy” and “impossible?”

Next, we build intuition based on informal argument
Sampling Theory (informal)

Exact reconstruction requires constraint on the rate at which $i(x,y)$ can change between samples

- “rate of change” means derivative
- the formal concept is **bandlimited signal**
- “bandlimit” and “constraint on derivative” are linked

Think of music

- bandlimited if it has some maximum **temporal frequency**
- the upper limit of human hearing is about 20 kHz

Think of imaging systems. Resolving power is measured in

- “line pairs per mm” (for a bar test pattern)
- “cycles per mm” (for a sine wave test pattern)

An image is bandlimited if it has some maximum **spatial frequency**
Example: A Simple Sine Wave

How do we discretize the signal?
Example: A Simple Sine Wave

How do we discretize the signal?
Example: A Simple Sine Wave

How do we discretize the signal?

How many samples should I take?
Can I take as many samples as I want?
Example: A Simple Sine Wave

How do we discretize the signal?

How many samples should I take?

Can I take as few samples as I want?

Slide Credit: Ioannis (Yannis) Gkioulkas (CMU)
**Example**: A Simple Sine Wave

How do we discretize the signal?

![Sine Wave Diagram]

Signal can be confused with one at lower frequency

*Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)*
Example: A Simple Sine Wave

How do we discretize the signal?

Signal can be confused with one at lower frequency

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)
Example: A Simple Sine Wave

How do we discretize the signal?

Signal can always be confused with one at higher frequency

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)
Undersampling = Aliasing
The challenge to intuition is the fact that music (in the 1D case) and images (in the 2D case) can be represented as linear combinations of individual sine waves of differing frequencies and phases (remember discussion on FFTs).

A fundamental result (**Sampling Theorem**) is:

For bandlimited signals, if you sample regularly at or above twice the maximum frequency (called the **Nyquist rate**), then you can reconstruct the original signal exactly.
Question: For a bandlimited signal, what if you oversample (i.e., sample at greater than the Nyquist rate)
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Answer: Nothing bad happens! Samples are redundant and there are wasted bits
**Question**: For a bandlimited signal, what if you **oversample** (i.e., sample at greater than the Nyquist rate)?

**Answer**: Nothing bad happens! Samples are redundant and there are wasted bits.

**Question**: For a bandlimited signal, what if you **undersample** (i.e., sample at less than the Nyquist rate)?
Sampling Theory (informal)

**Question:** For a bandlimited signal, what if you **oversample** (i.e., sample at greater than the Nyquist rate)

**Answer:** Nothing bad happens! Samples are redundant and there are wasted bits

**Question:** For a bandlimited signal, what if you **undersample** (i.e., sample at less than the Nyquist rate)

**Answer:** Two bad things happen! Things are missing (i.e., things that should be there aren’t) There are artifacts (i.e., things that shouldn’t be there are)