Lecture 7: Sampling

(unless otherwise stated slides are taken or adopted from Bob Woodham, Jim Little, Leon Sigal and Fred Tung)
Menu for Today (September 22, 2021)

Topics:
— Sampling
— Aliasing
— Bandlimited Signals
— Nyquist rate

Readings:
— Today’s Lecture: [Optional] Forsyth & Ponce (2nd ed.) 4.4
— Next Lecture: Forsyth & Ponce (2nd ed.) 4.5

Reminders:
— Assignment 1: Image Filtering and Hybrid Images due September 27

QUIZ – Sep 23, on Canvas – see message in Piazza
Today’s “fun” Example: Clever Hans

Hans could get 89% of the math questions right
Today’s “fun” Example: **Clever** Hans

Hans could get 89% of the math questions right.

The horse was **smart**, just not in the way van Osten thought!
Clever DNN
Visual Question Answering

Is there zebra climbing the tree?

Yes
Lecture 6: Re-cap

Convolution is **associative** and **symmetric** (correlation is not in general)

Convolution of a Gaussian with a Gaussian is another Gaussian

The **median filter** is a non-linear filter that selects the median in the neighbourhood

**Bilateral filter** is edge-preserving and non-linear. It is expressed as a product of **domain** and **range** kernels.
Framework for Today’s Topic

**Problem:** How do we go from the optics of image formation to digital images as arrays of numbers?

**Key Idea(s):** Sampling and the notion of bandlimited functions

**Theory:** Sampling Theory
Images are a **discrete**, or **sampled**, representation of a continuous world
What is an **Image**?

Up to now provided a **physical characterization**
— image formation as a problem in physics/optics
— we also talked about simple image processing algorithms on image arrays

Now provide a **mathematical characterization**
— to understand how to represent images digitally
— to understand how to compute with images
Where are the concepts we learn today will be useful?

**Image Formation**

Continuous (world) -> Discrete (image)
Where are the concepts we learn today will be useful?

Image **Formation**

Continuous (world) -> Discrete (image)

Image / Aspect Ratio **Resizing**

Discrete (image) -> Discrete (image)
Continuous Case

“Image” suggests a 2D surface whose appearance varies from point-to-point — the surface typically is a plane (but might be curved, e.g., as it is with an eye)

Appearance can be **Greyscale** (Black and White) or **Colour**

In **Greyscale**, variation in appearance can be described by a single parameter corresponding to the amount of light reaching the image at a given point in a given time
Continuous Case

Denote the image as a function, $i(x, y)$, where $x$ and $y$ are spatial variables.

**Aside**: The convention for this section is to use lower case letters for the continuous case and upper case letters for the discrete case.
Continuous Case: Observations

— $i(x, y)$ is a real-valued function of real spatial variables, $x$ and $y$
Recall: Pinhole Camera

Forsyth & Ponce (2nd ed.) Figure 1.2
Continuous Case: Observations

— $i(x, y)$ is a **real-valued function** of **real spatial variables**, $x$ and $y$

— $i(x, y)$ is **bounded above and below**. That is

$$0 \leq i(x, y) \leq M$$

for some maximum brightness $M$
Continuous Case: Observations

— \( i(x, y) \) is a real-valued function of real spatial variables, \( x \) and \( y \)

— \( i(x, y) \) is bounded above and below. That is

\[
0 \leq i(x, y) \leq M
\]

for some maximum brightness \( M \)

— \( i(x, y) \) is bounded in extent. That is, \( i(x, y) \) has a value over, at most, a bounded region
Continuous Case

— Images also can be considered a function of time. Then, we write \( i(x, y, t) \) where \( x \) and \( y \) are spatial variables and \( t \) is a temporal variable.

— To make the dependence of brightness on wavelength explicit, we can instead write \( i(x, y, t, \lambda) \) where \( x, y \) and \( t \) are as above and where \( \lambda \) is a spectral variable (wavelength).

— More commonly, we think of “colour” already as discrete and write

\[
\begin{align*}
  i_R(x, y) \\
  i_G(x, y) \\
  i_B(x, y)
\end{align*}
\]

for specific colour channels, R, G and B.
**Discrete Case**

**Idea:** Superimpose (regular) grid on continuous image

Sample the underlying continuous image according to the **tessellation** or **tiling** imposed by the grid
Discrete Case
**Discrete Case**

Each grid cell is called a picture element (**pixel**)

![Image of a grid with pixel notation](image.png)

Denote the discrete image as \( I(X, Y) \)

We can store the pixels in a matrix or array
Discrete Case

**Question:** How to sample?
— Sample brightness at the point?
— “Average” brightness over entire pixel?

**Answer:**
— Point sampling is useful for theoretical development
— Area-based sampling occurs in practice
**Discrete Case**

**Question**: What about the brightness samples themselves?
Discrete Case

**Question:** What about the brightness samples themselves?

**Answer:** We make values of $I(X, Y)$ discrete as well

Recall: $0 \leq i(x, y) \leq M$

We divide the range $[0, M]$ into a finite number of equivalence classes. This is called **quantization**.

The values are called **grey-levels**.
Discrete Case

Quantization is a topic in its own right

For now, a simple linear scheme is sufficient

Suppose $n$ bits-per-pixel are available. One can divide the range $[0, M]$ into evenly spaced intervals as follows:

$$i(x, y) \rightarrow \left\lfloor \frac{i(x, y)}{M} (2^n - 1) + 0.5 \right\rfloor$$

where $\lfloor \cdot \rfloor$ is floor (i.e., greatest integer less than or equal to)

Typically $n = 8$ resulting in grey-levels in the range $[0, 255]$
It is clear that *some* information may be lost when we work on a discrete pixel grid.
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Sampling

It is clear that \textit{some} information may be lost when we work on a discrete pixel grid.

Forsyth & Ponce (2nd ed.) Figure 4.7
Sampling

It is clear that some information may be lost when we work on a discrete pixel grid.

Forsyth & Ponce (2nd ed.) Figure 4.7
Question: When is $I(X, Y)$ an exact characterization of $i(x, y)$?
**Sampling Theory (informal)**

**Question**: When is $I(X, Y)$ an exact characterization of $i(x, y)$?

**Question (modified)**: When can we reconstruct $i(x, y)$ exactly from $I(X, Y)$?
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**Question** (modified): When can we reconstruct $i(x, y)$ exactly from $I(X, Y)$?

**Intuition**: Reconstruction involves some kind of interpolation (reconstructing a continuous function from discrete points)
Question: When is $I(X, Y)$ an exact characterization of $i(x, y)$?

Question (modified): When can we reconstruct $i(x, y)$ exactly from $I(X, Y)$?

Intuition: Reconstruction involves some kind of interpolation

Heuristic: When in doubt, consider simple cases (1D)
**Sampling Theory (informal)**

**Case 0:** Suppose \( i(x) = k \) (with \( k \) being one of our grey levels)

**Note:** we use equidistant sampling at integer values for convenience, in general, sampling doesn’t need to be equidistant
Sampling Theory (informal)

Case 0: Suppose \( i(x) = k \) (with \( k \) being one of our gray levels)

This is easy!
Sampling Theory (informal)

**Case 0**: Suppose \( i(x) = k \) (with \( k \) being one of our gray levels).

\[ l(X) = k \]. Any standard interpolation function would give \( i(x) = k \) for non-integer \( x \) (irrespective of how coarse the sampling is).
**Sampling Theory (informal)**

**Case 1:** Suppose \( i(x) = k \) has a discontinuity not falling precisely at integer \( x \).

Both \( k \)s are integer (quantized) values and so are exactly represented.
**Sampling Theory (informal)**

**Case 1**: Suppose $i(x)=k$ has a discontinuity not falling precisely at integer $x$.

We cannot reconstruct $i(x)$ exactly because we can never know exactly where the discontinuity lies.
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We cannot reconstruct \( i(x) \) exactly because we can never know exactly where the discontinuity lies.

This is **impossible!**
Question: How do we close the gap between “easy” and “impossible?”

Next, we build intuition based on informal argument.
**Sampling Theory (informal)**

Exact reconstruction requires constraint on the rate at which \( i(x,y) \) can change between samples

— “rate of change” means derivative
— the formal concept is **bandlimited signal**
— “bandlimit” and “constraint on derivative” are linked

Think of music

— bandlimited if it has some maximum **temporal frequency**
— the upper limit of human hearing is about 20 kHz

Think of imaging systems. Resolving power is measured in

— “line pairs per mm” (for a bar test pattern)
— “cycles per mm” (for a sine wave test pattern)

An image is bandlimited if it has some maximum **spatial frequency**
Example: A Simple Sine Wave

How do we discretize the signal?
Example: A Simple Sine Wave

How do we discretize the signal?
Example: A Simple Sine Wave

How do we discretize the signal?

How many samples should I take?
Can I take as many samples as I want?

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)
Example: A Simple Sine Wave

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Example: A Simple Sine Wave

How do we discretize the signal?

Signal can be confused with one at lower frequency.
Example: A Simple Sine Wave

How do we discretize the signal?

A signal can be confused with one at lower frequency
**Example:** A Simple Sine Wave

How do we discretize the signal?

A signal can always be confused with one at higher frequency
Undersampling = Aliasing
Sampling Theory (informal)

The challenge to intuition is the fact that music (in the 1D case) and images (in the 2D case) can be represented as linear combinations of individual sine waves of differing frequencies and phases (remember discussion on Fourier Transforms)

A fundamental result (Sampling Theorem) is:

For bandlimited signals, if you sample regularly at or above twice the maximum frequency (called the Nyquist rate), then you can reconstruct the original signal exactly
**Sampling Theory (informal)**

**Question**: For a bandlimited signal, what if you **oversample** (i.e., sample at greater than the Nyquist rate)
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**Answer**: Nothing bad happens! Samples are redundant and there are wasted bits
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**Question**: For a bandlimited signal, what if you **undersample** (i.e., sample at less than the Nyquist rate)

**Answer**: Two bad things happen! Things are missing (i.e., things that should be there aren’t). There are artifacts (i.e., things that shouldn’t be there are)
Sampling Theory (informal)

Forsyth & Ponce (2nd ed.) Figure 4.7
Sampling Theory (informal)

Forsyth & Ponce (2nd ed.) Figure 4.12

Product of sinusoids varying in frequency. Then resampled without smoothing, by a factor of two. Scaled to same size.
Reducing Aliasing Artifacts

1. **Oversampling** — sample more than you think you need and average (i.e., area sampling)
Aliasing

- aliasing artifacts
- anti-aliasing by oversampling

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)
Reducing Aliasing Artifacts

1. **Oversampling** — sample more than you think you need and average (i.e., area sampling)

2. **Smoothing** before sampling. Why?
Aliasing in Photographs

This is also known as “moire”

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)
Temporal Aliasing

Imagine a spoked wheel moving to the right (rotating clockwise). Mark wheel with dot so we can see what’s happening.

If camera shutter is only open for a fraction of a frame time (frame time = 1/30 sec. for video, 1/24 sec. for film):

Without dot, wheel appears to be rotating slowly backwards! (counterclockwise)
**Temporal Aliasing**

**Wagon wheel effect**

*Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)*
Temporal Aliasing

Slide Credit: Ioannis (Yannis) Gkioulekas (CMU)
**Sampling Theory (informal)**

Sometimes **undersampling** is unavoidable, and there is a trade-off between “things missing” and “artifacts.”

— **Medical imaging**: usually try to maximize information content, tolerate some artifacts

— **Computer graphics**: usually try to minimize artifacts, tolerate some information missing in order to be visually pleasing