CPSC 505 Example: Laplacian vs Second Directional Derivative

Our treatment of edge detection in class has focused on the need to "regularize" (i.e., to make well-posed) the differentiation step. To complete this treatment, we re-examine the differentiation step to consider another possible second derivative operator to use.

As we have seen, one can characterize sharp intensity changes by:

1. extrema of a first-order derivative operator
2. zero-crossings of a second-order derivative operator

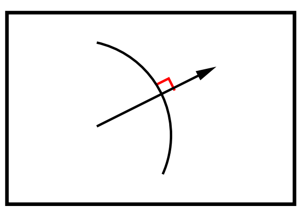
Further things to consider are:

1. linear versus non-linear operators
2. directional versus rotationally invariant operators

Aside: In $\Re^2$, the first derivative in any direction can be expressed as a linear combination of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Even so, estimating first derivatives in more than two directions can be helpful in the presence of noise (i.e., to improve the SNR).

Recall: The Marr/Hildreth choice of the Laplacian was motivated, in part, by the desire for a linear, rotationally invariant, second derivative operator.

The second directional derivative along the gradient of a function, $f(x, y)$, is the (one-dimensional) second derivative taken in the direction of maximum change of the function. A diagram may help.



← A more natural definition of edge?

Consider a point (x, y). The curve in the figure represents an iso-contour (i.e., a contour of constant value) of the function, $f(x, y)$, at (x, y). The direction of maximum change (the straight line with the arrow) necessarily is orthogonal to the iso-contour at (x, y). The second directional derivative along the gradient at (x, y) is the second derivative along this line.

Observations regarding the Laplacian and the second directional derivative along the gradient:

1. Both are second-order derivative operators
2. Both are rotationally invariant

Aside: The fact that the second directional derivative along the gradient is rotationally invariant is not obvious.

Recall that the Laplacian, $\nabla^2 f(x, y)$, is defined by

$$\nabla^2 f(x, y) \equiv \frac{\partial^2 f(x, y)}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2}$$

Let's switch instead to a simplified notation:

$$\nabla^2 f \;\equiv\; f_{xx} \,+\, f_{yy}$$

where subscripts denote partial differentiation. Using this simplified notation, the second directional derivative along the gradient can be written as

$$\frac{\partial^2 f}{\partial n^2} \;\equiv\; \frac{{f_x}^2 f_{xx} \,+\, 2 f_x f_y f_{xy} \,+\, {f_y}^2 f_{yy}}{{f_x}^2 \,+\, {f_y}^2}$$

Wherein lies the difference? In mathematical terms

1. $\dfrac{\partial^2}{\partial n^2}$ is non-linear
2. $\dfrac{\partial^2}{\partial n^2}$ neither commutes nor associates with convolution

$$\frac{\partial^2}{\partial n^2}\,(g * f) \;\neq\; \left(\frac{\partial^2 g}{\partial n^2}\right) * f$$
$$\left(\frac{\partial^2 g}{\partial n^2}\right) * f \;\neq\; g * \left(\frac{\partial^2 f}{\partial n^2}\right)$$

3. $\dfrac{\partial^2}{\partial n^2}$ is not everywhere defined (i.e., we require ${f_x}^2 \,+\, {f_y}^2 \,\neq\, 0$)

When used as the second derivative operator for zero crossing detection

4. $\dfrac{\partial^2}{\partial n^2}$ provides better localization, especially at corners

There are many classes of function, f, for which the zero crossings of $\nabla^2 f$ and $\dfrac{\partial^2 f}{\partial n^2}$ are equivalent.

If ${f_x}^2 \,+\, {f_y}^2 \,\neq\, 0$, the zeros of $\dfrac{\partial^2 f}{\partial n^2}$ coincide with the zeros of $\nabla^2 f$ if and only if the mean curvature, H, is zero where

$$H \;\equiv\; \frac{\left(1 + {f_x}^2\right) f_{yy} \,+\, \left(1 + {f_y}^2\right) f_{xx} \,-\, 2 f_x f_y f_{xy}}{2\left(1 \,+\, {f_x}^2 \,+\, {f_y}^2\right)^3}$$

Aside: Torre & Poggio (1986) noted that most stimuli used in psychological experiments have $H = 0$ so that it was impossible to distinguish between $\nabla^2 f$ and $\dfrac{\partial^2 f}{\partial n^2}$ based on existing experimental evidence.