Our treatment of edge detection in class has focused on the need to “regularize” (i.e., to make well-posed) the differentiation step. To complete this treatment, we re-examine the differentiation step to consider another possible second derivative operator to use.

As we have seen, one can characterize sharp intensity changes by:

1. extrema of a first-order derivative operator
2. zero-crossings of a second-order derivative operator

Further things to consider are:

1. linear versus non-linear operators
2. directional versus rotationally invariant operators

Aside: In $\mathbb{R}^2$, the first derivative in any direction can be expressed as a linear combination of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Even so, estimating first derivatives in more than two directions can be helpful in the presence of noise (i.e., to improve the SNR).

Recall: The Marr/Hildreth choice of the Laplacian was motivated, in part, by the desire for a linear, rotationally invariant, second derivative operator.

The second directional derivative along the gradient of a function, $f(x, y)$, is the (one-dimensional) second derivative taken in the direction of maximum change of the function. A diagram may help.

Consider a point $(x, y)$. The curve in the figure represents an iso-contour (i.e., a contour of constant value) of the function, $f(x, y)$, at $(x, y)$. The direction of maximum change (the straight line with the arrow) necessarily is orthogonal to the iso-contour at $(x, y)$. The second directional derivative along the gradient at $(x, y)$ is the second derivative along this line.

Observations regarding the Laplacian and the second directional derivative along the gradient:

1. Both are second-order derivative operators
2. Both are rotationally invariant

Aside: The fact that the second directional derivative along the gradient is rotationally invariant is not obvious.

Recall that the Laplacian, $\nabla^2 f(x, y)$, is defined by

$$\nabla^2 f(x, y) \equiv \frac{\partial^2 f(x, y)}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2}$$
Let’s switch instead to a simplified notation:

\[ \nabla^2 f \equiv f_{xx} + f_{yy} \]

where subscripts denote partial differentiation. Using this simplified notation, the second directional derivative along the gradient can be written as

\[
\frac{\partial^2 f}{\partial n^2} = \frac{f_x^2 f_{xx} + 2f_x f_y f_{xy} + f_y^2 f_{yy}}{f_x^2 + f_y^2}
\]

Wherein lies the difference? In mathematical terms

1. \( \frac{\partial^2}{\partial n^2} \) is non-linear
2. \( \frac{\partial^2}{\partial n^2} \) neither commutes nor associates with convolution
   \[
   \frac{\partial^2}{\partial n^2} (g*f) \neq \left( \frac{\partial^2 g}{\partial n^2} \right)*f
   \]
   \[
   \left( \frac{\partial^2 g}{\partial n^2} \right)*f \neq g*\left( \frac{\partial^2 f}{\partial n^2} \right)
   \]
3. \( \frac{\partial^2}{\partial n^2} \) is not everywhere defined (i.e., we require \( f_x^2 + f_y^2 \neq 0 \))

When used as the second derivative operator for zero crossing detection

4. \( \frac{\partial^2}{\partial n^2} \) provides better localization, especially at corners

There are many classes of function, \( f \), for which the zero crossings of \( \nabla^2 f \) and \( \frac{\partial^2 f}{\partial n^2} \) are equivalent.

If \( f_x^2 + f_y^2 \neq 0 \), the zeros of \( \frac{\partial^2 f}{\partial n^2} \) coincide with the zeros of \( \nabla^2 f \) if and only if the mean curvature, \( H \), is zero where

\[
H \equiv \frac{(1 + f_x^2) f_{yy} + (1 + f_y^2) f_{xx} - 2f_x f_y f_{xy}}{2(1 + f_x^2 + f_y^2)^3}
\]

Aside: Torre & Poggio (1986) noted that most stimuli used in psychological experiments have \( H = 0 \) so that it was impossible to distinguish between \( \nabla^2 f \) and \( \frac{\partial^2 f}{\partial n^2} \) based on existing experimental evidence.