Sorting

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Introduction

Reading:

- CLRS: "Sorting in Linear Time" 8
- GT: "Sorting, Sets, and Selection" 4.4-4.5

Motivation:

- Insertion sort is worst-case $O(n^2)$.
- Other algorithms are worst-case $\Theta(n \log n)$.
  - e.g. mergesort and heapsort
- Can we do better?
  - Use decision trees to model sorting algorithms.
  - Decisions trees have worst-case $\Omega(n \log n)$.
  - Go outside decision tree model to do better ($\Theta(n)$).

Comparison Sorts

Recall: Sorting

- Input: A sequence of n values $a_1, a_2, \ldots, a_n$.
- Output: A permutation $b_1, b_2, \ldots, b_n$ of $a_1, a_2, \ldots, a_n$ such that $b_1 \leq b_2 \leq \ldots \leq b_n$.
- Instance: $3, 8, 2, 5$.

In a comparison sort algorithm, the sorted order is determined by a sequence of comparisons between pairs of elements.

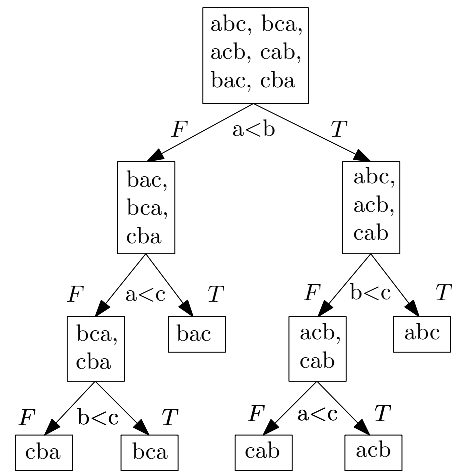
- Insertion sort, selection sort, bubble sort, quicksort, mergesort, and heapsort are comparison sorts.

Using a decision tree, we show that *every* comparison sort requires $\Omega(n \log n)$ comparisons in the worst-case.

Decision Tree

- Represents every sequence of comparisons that an algorithm might make on an input of size n.
- Nodes annotated with the orderings consistent with the comparisons made so far.
- Edges denote the result of a single comparison.
- Total order at leaves.

Algorithm: Insertion sort.
Instance ($n = 3$): the numbers a, b, c.



Lower Bound

Claim

The depth of a decision tree for a given value of n is $\Omega(n \log n)$.

Proof.

There are $n!$ leaves. A tree of height h has at most 2^{h+1} nodes. So

$$\begin{aligned} 2^{h+1} &\geq n! \\ h+1 &\geq \log_2 n! = \log_2(1 \cdot 2 \cdot \ldots \cdot n) \\ &= \log_2 1 + \log_2 2 + \ldots + \log_2 n \\ &> (n/2)\log_2(n/2) \\ h &\in \Omega(n \log n) \end{aligned}$$

□

Lower Bound (cont'd)

Theorem

Every comparison sort requires $\Omega(n \log n)$ comparisons in the worst-case.

Proof.

Given a comparison sort, we look at the decision tree it generates on a inputs of size n.

- Each path from root to leaf is one possible sequence of comparisons.
- Length of the path is the number of comparisons for that instance.
- Height of the tree is the worst-case path length (number of comparisons).

Height of the tree is $\Omega(n \log n)$ by the previous claim. Hence, every comparison sort requires $\Omega(n \log n)$ comparisons. □

Transitivity: Indirect Comparisons

- If $a < b$ and $b < c$, we indirectly know that $a < c$.
- Quicksort splits instance into sets A, C based on a pivot b.
  - A is such that $a \leq b$, for $a \in A$.
  - C is such that $b \leq c$, for $c \in C$.
  - So $a \leq c$, for $a \in A$ and $c \in C$ by transitivity.
- Algorithms doing better than $\Omega(n \log n)$ in the worst-case
  - Do not pivot on an element of the instance.
    - Escapes decision tree model.
  - Use knowledge of problem domain to choose pivot independent of particular instance.

Bucket Sort (Counting Sort)

Assume keys are integers in ranging from 0 to $N-1$.

- Pivots are $0, 1, \ldots, N-1$.
- One set (bucket) per possible key.

```
Algorithm BucketSort(A,N)
   Let S be an empty list
   Let B[0...N-1] be an array of empty lists
   for i ← 0 to A.length-1 do
      append A[i] to B[A[i].key]
   for j ← 0 to N-1 do
      for each element x of B[j] do      // in order
         append x to S
   return S
```

Time Complexity: $\Theta(n + N)$

Bucket Sort (cont'd)

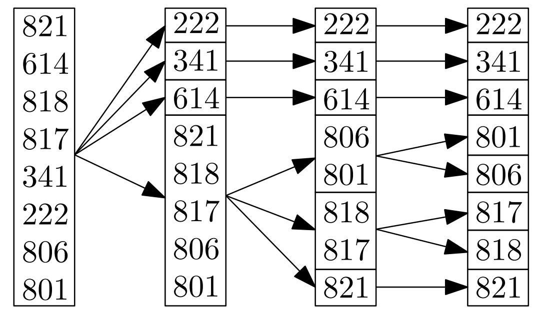
- In practice, we compact `B[0],...,B[N-1]` in one array by
  - precomputing their maximum sizes (offsets) and
  - using an array of their current lengths.
- Bucket sort is stable:
  - if `i < j` and `A[i].key = A[j].key` then `A[i]` comes before `A[j]` in `S`
- Stable sorts can sort a class list in two passes.
  - Sort the list by first name.
  - Then sort the list by last name.
- Which algorithms are stable?

Radix Sort

Intuitively, we sort n integers with at most d-digits by

- binning according to their most significant digit,
- sorting each pile recursively, and
  - i.e. split on the 2nd most significant digit, then 3rd most, etc.
- merging the results.

Too slow because there are too many piles.

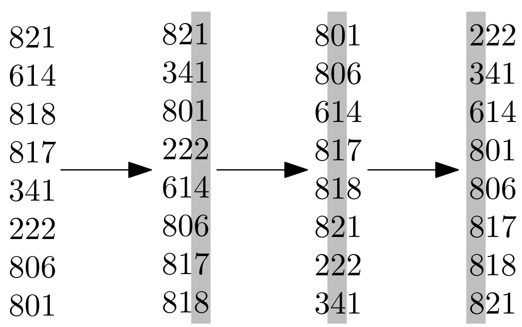


Radix Sort (cont'd)

Instead, we use a stable sort to get rid of the piles.

- Sort digit by digit, from the least significant digit to the most significant digit.

```
Algorithm RadixSort(A,d)
  for i ← 0 to d-1 do
    sort A on digit i using BucketSort
```



Radix Sort Correctness

Claim

After the ith iteration, the values are sorted by their last i digits.

Proof.

Induct on i. Trivially true when $i = 1$. So consider $i > 1$. Let x and y be numbers such that the

- the last i digits $x_i, x_{i-1}, \ldots, x_2, x_1$ of x are less than
- the last i digits $y_i, y_{i-1}, \ldots, y_2, y_1$ of y.

We need to show that x comes before y.

- If $x_i = y_i$, then $x_{i-1}, \ldots, x_2, x_1 < y_{i-1}, \ldots, y_2, y_1$. So x is ordered before y on the $(i-1)$th iteration. `BucketSort` preserves this order because it is stable.
- If $x_i < y_i$ then `BucketSort` orders x and y correctly on this iteration.

□

Radix Sort Complexity

We call `BucketSort` d times.

- `BucketSort` takes $\Theta(n + N)$ time.

So the complexity of `RadixSort` is $\Theta(d(n + N))$.

- `RadixSort` is $O(n)$, if the range of values is small
  - i.e. d is constant and $N \in O(n)$