Medians and Order Statistics

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Introduction

Reading:

- CLRS: "Medians and Order Statistics" 9
- GT: "Sorting, Sets, and Selection" 4.7

Problem

How can we find the i^{th} smallest element of an unsorted array? Sorting $\Theta(n \log n)$ will work, but we can do better.

Definition

The i^{th} order statistic of a set with n elements is the i^{th} smallest element of the set.

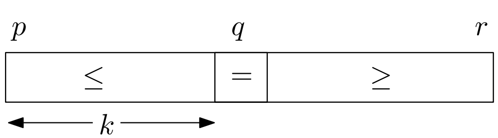
The min is the 1^{st} order statistic, the max is the n^{th} order statistic, and the median is the $\lfloor n/2 \rfloor^{th}$ order statistic.

5 8 13 [21] 34 55 89
2 4 8 [16] 32 64 128 256

QuickSort Review

```
Algorithm QuickSort(A,p,r)
   if p < r then
      q ← Partition(A,p,r);
      QuickSort(A,p,q − 1)
      QuickSort(A,q + 1,r)
```

Why sort both partitions for selection?



Deterministic Divide-and-Conquer

```
Algorithm Select(A,p,r,i)
   if (p = r) then
      return A[p];
   q ←Partition(A,p,r);
   k ← q − p;            // size of left side
   if (i = k + 1) then
      return A[q];
   else if (i ≤ k) then
      return Select(A,p,q − 1,i);
   else
      return Select(A,q + 1,r,i − (k + 1));
```

Deterministic Divide-and-Conquer (cont'd)

Worst-Case Complexity

- Look for the smallest.
- Always pivot around the largest element.
- Generates $\Theta\left(n^2\right)$ runtime.

Idea: Pick the pivot at random

Definition

An algorithm is randomized if its behaviour depends on the input and the values produced by a random number generator.

Randomized algorithms are nice when they are

- simple and
- have good average-case running times.

Randomized Divide-and-Conquer

Same worst case complexity because of bad guesses!

So randomly choose a good partition:

- Good if each side has at most $3n/4$ elements.
- $P[\text{partition is good}] \approx 1/2$ because every partition is equally likely.
- About two guesses to get a good partition, on average.
- So $O(n)$ time to get a good partition, on average.

Randomized Divide-and-Conquer (cont'd)

Worst case: Element is partition with at most $3n/4$ elements.

$$T(n) = \begin{cases} T(3n/4) + O(n) & \text{if } n \geq 0 \\ O(1) & \text{otherwise} \end{cases}$$

Case 3 Master Theorem: $T(n) \in O(n)$.

Not true average because we assumed element in large partition.

Deterministically Finding a Pivot

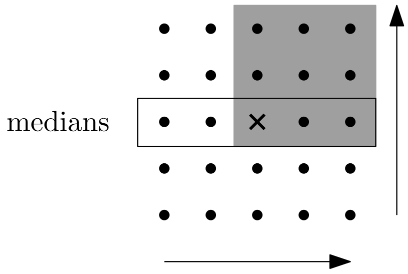
Must efficiently choose a guaranteed good pivot.

- Finding median too hard.
- Testing a constant number of elements too "local".
- Elect samples to represent fixed-size subgroups.
- Recursively elect leader (pivot) from representatives.

Specifically,

- Each group of five elects representative (median).
- Recursively call selection to find the median of the medians.

Pivot quality



How many elements must be greater or equal to the pivot?

At least 3 for every representative greater than our pivot.

$$3\lfloor\lfloor n/5\rfloor/2\rfloor \geq \text{pivot}$$

$$\begin{aligned}
3\lfloor\lfloor n/5\rfloor/2\rfloor &= 3\lfloor n/10\rfloor \geq 3\left[n/10-1\right] \\
&= 3n/10-3 \\
&= 6n/20-3=(6n-60)/20 \\
&= (5n+n-60)/20 \geq 5n/20, \text{ if } n \geq 60 \\
&= n/4
\end{aligned}$$

As n gets large, we eliminate at least one quarter of the elements.

Worst case complexity

Our worst case recurrence is

$$T(n) = \begin{cases} T\left(\lfloor n/5\rfloor\right) + T\left(3n/4\right) + dn & \text{for } n > 60 \\ \Theta(1) & \text{otherwise} \end{cases}$$

We guess that $T(n) \leq cn$. Try the inductive step.

$$\begin{aligned}
T(n) &\leq T\left(\lfloor n/5\rfloor\right) + T(3n/4) + dn \\
&\leq c\lfloor n/5\rfloor + c\cdot 3n/4 + dn \text{ by inductive hypothesis} \\
&\leq c\cdot n/5 + c\cdot 3n/4 + dn \\
&= c\cdot 19n/20 + dn \\
&\leq cn, \text{ if } n \geq 20d
\end{aligned}$$

Only 60 base cases, so we can choose c large enough!