Master Method

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Introduction

Reading:

- CLRS "Recurrences" 4.3
- GT "Divide-and-conquer" 5.2.1, 5.2.2

We state the Master Theorem, which gives a tight asymptotic bound on a large class of recurrence relations of the form $T(n) = aT(n/b) + f(n)$.

Then we quickly analyse the run-time of two arithmetic operations.

The Master Method

Theorem

Let $a \geq 1, b > 1$ be constants. Let $T(n)$ be defined by $T(n) = aT(n/b) + f(n)$, where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then

1. *If $f(n) \in O\left(n^{(\log_b a)-\epsilon}\right)$ for some $\epsilon > 0$, then $T(n) \in \Theta\left(n^{\log_b a}\right)$.*
2. *If $f(n) \in \Theta\left(n^{\log_b a}\right)$, then $T(n) \in \Theta\left(n^{\log_b a} \cdot \log n\right)$.*
3. *If $f(n) \in \Omega\left(n^{(\log_b a)+\epsilon}\right)$ for some $\epsilon > 0$ and $af(n/b) \leq \delta f(n)$ for some $\delta < 1$ and n sufficiently large, then $T(n) \in \Theta(f(n))$.*

Intuitively,

1. Leaves dominate.
2. Costs are balanced.
3. Internal nodes dominate, but do not explode.

Multiplying Large Integers

Problem

Given two k-digit integers in base 10 (or 2^i), compute their product.

Method 1: Long multiplication

$$
\begin{array}{r}
326 \\
\times \quad 57 \\
\hline
2282 \\
1630 \\
\hline
18582
\end{array}
$$

Multiply each digit of n_1 by each digit of n_2 and add up the digits in each position. So $\Theta\left(k^2\right)$.

Method 2: Divide-and-conquer

Divide each number into two numbers with at most $\lceil k/2 \rceil$ digits. For example,

$$
\begin{array}{llll}
n_1 = 31|28 & a = 31 & b = 28 & n_1 = a \cdot 10^{\lfloor k/2 \rfloor} + b \\
n_2 = 17|93 & c = 17 & d = 93 & n_2 = c \cdot 10^{\lfloor k/2 \rfloor} + d
\end{array}
$$

Multiplying Large Integers (cont'd)

Method 2 (cont'd)

Multiply numbers

$$\begin{aligned} n_1 \cdot n_2 &= (ax + b) \cdot (cx + d), \text{ where } x = 10^{\lfloor k/2 \rfloor} \\ &= acx^2 + adx + bcx + bd \\ &= acx^2 + (ad + bc)x + bd \end{aligned}$$

Multiplying by x is $\Theta(k)$. Additions and subtractions are $\Theta(k)$.

$$T(k) = \begin{cases} 4T(k/2) + \Theta(k) & \text{for } k \geq 2 \\ \Theta(1) & k = 1 \end{cases}$$

By the Master Theorem $T(k) \in \Theta\left(n^{\log_2 4}\right) = \Theta\left(n^2\right)$.

Multiplying Large Integers (cont'd)

Method 2 (cont'd)

Idea: Reuse ac, bd to reduce multiplication.

$$\begin{aligned}
n_1 \cdot n_2 = &\ acx^2 + (ad + bc)x + bd \\
= &\ acx^2 + (ad + bc + [ac - ac] + [bd - bd])x + bd \\
= &\ acx^2 + ([ad + ac] + [bc + db] - ac - bd)x + bd \\
= &\ acx^2 + (a[c + d] + b[c + d] - ac - bd)x + bd \\
= &\ acx^2 + ([a + b][c + d] - ac - bd)x + bd
\end{aligned}$$

Then

$$T(k) = \begin{cases} 3T(k/2) + \Theta(k) & \text{for } k \geq 2 \\ \Theta(1) & k = 1 \end{cases}$$

By the Master Theorem $T(k) \in \Theta\left(n^{\log_2 3}\right) \approx \Theta\left(n^{1.585}\right)$.

Evaluating Large Exponents

Problem

Given an n-digit integer a and an integer b, compute a^b.

Method 1: Iteration

Expand a^b to $a \times a \times \ldots \times a$.

Last multiplication is $\Theta(bn)$ digits times n digits.

- Long multiplication is $O\left(bn^2\right)$.
- Fast multiplication is $O\left(b^{1.58}n^{1.58}\right)$.

If $b \geq n$ then long multiplication is faster.

So we use long multiplication.

- $b/2$ multiplications multiply more than $b/2 \times n$ digits by a n digits number.

Running time is $\Omega\left(b^2n^2\right)$

Evaluating Large Exponents (cont'd)

Method 2: Divide-and-conquer

We compute a^b recursively

$$a^b = \begin{cases} a^{\lfloor b/2 \rfloor} \times a^{\lfloor b/2 \rfloor} & \text{if } b \text{ is even} \\ a^{\lfloor b/2 \rfloor} \times a^{\lfloor b/2 \rfloor} \times a & \text{if } b \text{ is odd} \end{cases}$$

Each iteration multiplies bn digits by bn digits.

$$T(b) = T(b/2) + \Theta\left(b^{1.58} n^{1.58}\right)$$

Case 3 of the Master Theorem because $b^{1.58} \in \Omega\left(b^{(\log_2 1)+\epsilon}\right)$.
Check regularity

$$c \cdot (b/2)^{1.58} n^{1.58} < \delta \times c \cdot b^{1.58} n^{1.58}$$

So $T(n) \in \Theta\left(b^{1.58} n^{1.58}\right)$