

Master Method

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Introduction

Reading:

- ▶ CLRS “Recurrences” 4.3
- ▶ GT “Divide-and-conquer” 5.2.1, 5.2.2

We state the Master Theorem, which gives a tight asymptotic bound on a large class of recurrence relations of the form

$$T(n) = aT(n/b) + f(n).$$

Then we quickly analyse the run-time of two arithmetic operations.

The Master Method

Theorem

Let $a \geq 1, b > 1$ be constants. Let $T(n)$ be defined by $T(n) = aT(n/b) + f(n)$, where n/b means either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then

1. If $f(n) \in O(n^{(\log_b a) - \epsilon})$ for some $\epsilon > 0$, then $T(n) \in \Theta(n^{\log_b a})$.
2. If $f(n) \in \Theta(n^{\log_b a})$, then $T(n) \in \Theta(n^{\log_b a} \cdot \log n)$.
3. If $f(n) \in \Omega(n^{(\log_b a) + \epsilon})$ for some $\epsilon > 0$ **and** $af(n/b) \leq \delta f(n)$ for some $\delta < 1$ and n sufficiently large, then $T(n) \in \Theta(f(n))$.

Intuitively,

1. Leaves dominate.
2. Costs are balanced.
3. Internal nodes dominate, but do not explode.

Multiplying Large Integers

Problem

Given two k -digit integers in base 10 (or 2^i), compute their product.

Method 1: Long multiplication

$$\begin{array}{r} 326 \\ \times 57 \\ \hline 2282 \\ 1630 \\ \hline 18582 \end{array}$$

Multiply each digit of n_1 by each digit of n_2 and add up the digits in each position. So $\Theta(k^2)$.

Method 2: Divide-and-conquer

Divide each number into two numbers with at most $\lceil k/2 \rceil$ digits. For example,

$$\begin{array}{llll} n_1 = 31 | 28 & a = 31 & b = 28 & n_1 = a \cdot 10^{\lfloor k/2 \rfloor} + b \\ n_2 = 17 | 93 & c = 17 & d = 93 & n_2 = c \cdot 10^{\lfloor k/2 \rfloor} + d \end{array}$$

Multiplying Large Integers (cont'd)

Method 2 (cont'd)

Multiply numbers

$$\begin{aligned}n_1 \cdot n_2 &= (ax + b) \cdot (cx + d), \text{ where } x = 10^{\lfloor k/2 \rfloor} \\&= acx^2 + adx + bcx + bd \\&= acx^2 + (ad + bc)x + bd\end{aligned}$$

Multiplying by x is $\Theta(k)$. Additions and subtractions are $\Theta(k)$.

$$T(k) = \begin{cases} 4T(k/2) + \Theta(k) & \text{for } k \geq 2 \\ \Theta(1) & k = 1 \end{cases}$$

By the Master Theorem $T(k) \in \Theta(n^{\log_2 4}) = \Theta(n^2)$.

Multiplying Large Integers (cont'd)

Method 2 (cont'd)

Idea: Reuse ac, bd to reduce multiplication.

$$\begin{aligned}n_1 \cdot n_2 &= acx^2 + (ad + bc)x + bd \\&= acx^2 + (ad + bc + [ac - ac] + [bd - bd])x + bd \\&= acx^2 + ([ad + ac] + [bc + db] - ac - bd)x + bd \\&= acx^2 + (a[c + d] + b[c + d] - ac - bd)x + bd \\&= acx^2 + ([a + b][c + d] - ac - bd)x + bd\end{aligned}$$

Then

$$T(k) = \begin{cases} 3T(k/2) + \Theta(k) & \text{for } k \geq 2 \\ \Theta(1) & k = 1 \end{cases}$$

By the Master Theorem $T(k) \in \Theta(n^{\log_2 3}) \approx \Theta(n^{1.585})$.

Evaluating Large Exponents

Problem

Given an n -digit integer a and an integer b , compute a^b .

Method 1: Iteration

Expand a^b to $a \times a \times \dots \times a$.

Last multiplication is $\Theta(bn)$ digits times n digits.

- ▶ Long multiplication is $O(bn^2)$.
- ▶ Fast multiplication is $O(b^{1.58}n^{1.58})$.

If $b \geq n$ then long multiplication is faster.

So we use long multiplication.

- ▶ $b/2$ multiplications multiply more than $b/2 \times n$ digits by a n digits number.

Running time is $\Omega(b^2n^2)$

Evaluating Large Exponents (cont'd)

Method 2: Divide-and-conquer

We compute a^b recursively

$$a^b = \begin{cases} a^{\lfloor b/2 \rfloor} \times a^{\lfloor b/2 \rfloor} & \text{if } b \text{ is even} \\ a^{\lfloor b/2 \rfloor} \times a^{\lfloor b/2 \rfloor} \times a & \text{if } b \text{ is odd} \end{cases}$$

Each iteration multiplies bn digits by bn digits.

$$T(b) = T(b/2) + \Theta(b^{1.58}n^{1.58})$$

Case 3 of the Master Theorem because $b^{1.58} \in \Omega(b^{(\log_2 1)+\epsilon})$.

Check regularity

$$c \cdot (b/2)^{1.58}n^{1.58} < \delta \times c \cdot b^{1.58}n^{1.58}$$

So $T(n) \in \Theta(b^{1.58}n^{1.58})$