# Dynamic Programming

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# Introduction

**Reading:**

- "Dynamic Programming", 15 CLRS
- "Dynamic Programming", 5.3 GT

Greedy works if some optimal solution contains the greedy choice.

- Dijkstra's algorithm always adds the cheapest vertex to the shortest path tree (greedy).
- Dijkstra's algorithm may not work with negative edge weights.

Dynamic programming tries all possible choices.

- Bellman-Ford's algorithm attempts every one edge extension of shortest paths (exhaustive).
- Bellman-Ford's algorithm works with negative edge weights.

# Optimal substructure

Both Dijkstra's and Bellman-Ford's algorithms work because you can extend the optimal solution of a subproblem.

**Definition**

A problem has optimal substructure if some optimal solution is

- an optimal solution to a subproblem combined with
- an optimal choice.

- Often don't know which choice to make, so try them all.
- May be efficient if subproblems overlap
  - $|V|$ paths to extend in Bellman-Ford (subproblems)
  - $|E|$ edges to extend with (choices)

# Making change with coins

## Problem

**Given**: Coin values $c_1, c_2, \ldots, c_t$ with which to make change and the amount of change to be made $n$.

**Wanted**: Number of each coin to use $n_1, n_2, \ldots, n_t$ such that sum of coins is $n$ and fewest coins are used.

Denominations chosen so that greedy algorithm works, but not true in general.

## Example

| | | | |
|---|---|---|---|
| Coins: 1¢, 3¢, and 4¢ | Greedy | $\longrightarrow$ | 4¢, 1¢, and 1¢ |
| Change to make: 6¢ | Optimal | $\longrightarrow$ | 3¢ and 3¢ |

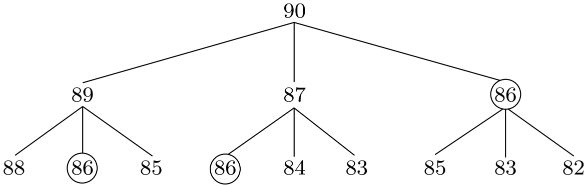
# Exhaustive coin changing

```
Algorithm TryEmAll(C,n)
    int N[C.length]
    for i ← 0 to N.length−1 do
        N[i] ← 0
    if n = 0 then
        return N
    N[1] ← ∞
    for i ← 0 to C.length−1 do
        if n ≥ C[i] then
            subprob ← TryEmAll(C,n − C[i])
            if subprob.sum()+1 < N.sum() then
                N ← subprob
                N[i] ← N[i] + 1
    return N
```

# Recursion tree for `TryEmAll`

This is inefficient because it recomputes the same subproblems over and over again.

`TryEmAll([1,3,4],90)`




A better idea: replace each recursive call with a table look-up.

- Construct a table to store the optimal solution for each $n$.
- Iteratively increase $n$ and compute its entry.

# Dynamic programming solution

```
Algorithm DPCoinChange(C,n)
    int N[n+1][C.length]

    for i ← 0 to C.length−1 do
        N[0][i] ← 0

    for m ← 1 to n do
        N[m][0] ← ∞
        for i ← 0 to C.length−1 do
            if m ≥ C[i] then
                if N[m − C[i]].sum()+1 < N[m].sum() then
                    N[m] ← N[m − C[i]]
                    N[m][i] ← N[m][i] + 1

    return N[n]
```

# Runtime complexity

What is the runtime complexity of this algorithm?

- If it updates $N[m]$ every time, then $n \times t$ updates.
- Each update copies $t$ integers.
- So $O(nt^2)$.

Advantages of eliminating the recursion:

- Counting argument for runtime complexity.
- No call stack overhead!

Why copy solution during update when we only chose one coin?

- Faster to remember the optimal choice and
- backtrack to recover the solution.

# Faster solution

```
Algorithm FastDPCoinChange(C,n)
    int minCoins[n+1], bestChoice[n+1]
    minCoins[0] ← 0
    bestChoice[0] ← −1
    // main loop
    for m ← 1 to n do
        minCoins[0] ← ∞
        for i ← 0 to C.length−1 do
            if m ≥ C[i] then
                if minCoins[m−C[i]]+1 < minCoins[m] then
                    minCoins[m] ← minCoins[m − C[i]] + 1
                    bestChoice[m] ← i
```

# Backtracking

```
// backtracking
int N[C.length]
for i ← 0 to N.length−1 do
    N[i] ← 0
while n > 0  do
    N[bestChoice[n]] ← N[bestChoice[n]] + 1
    n ← n − C[bestChoice[n]]
return N
```

Eliminates copying and adding $t$ integers from the innermost loop.

- Total runtime complexity is $O(nt)$.

**Example: 1¢, 3¢, and 4¢ coins**

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| N | 0 | 1 | 2 | 1 | 1 | 2 | 2 | 2 | 2 | | |
| bC | $\emptyset$ | 1 | 1 | 3 | 4 | 1 | 3 | 3 | 4 | | |

How do you make 9¢ and 10¢ change?

# Designing a dynamic programming algorithm

Decide on the parameters the problem will have.

- This gives the "shape" of the table and determines the runtime complexity.
- `FastDPCoinChange` only used $n$ to determine *bestChoice*, so table is an array.
- Limited supply of coins
  - Solution depends on $n$ and number of each type of coin available $(a_1, a_2, \ldots, a_t)$.
  - Table has one dimension for $n$, another for $a_1$, another for $a_2$, etc.

What do we need to store in the table:

- `DPCoinChange` stored all of the best choices made so far.
- `FastDPCoinChange` just stored the last best choice.

# Design (cont'd)

Express the problem in terms of smaller problems.

- `FastDPCoinChange`

$$minCoins[n] = \min\{1 + minCoins[n - C[i]] : C[i] \leq n\}$$

Determine how to fill-in the table.

- A subproblem solution must be computed before those that rely on it.
- Trickier for multi-dimensional tables. Typically row-by-row, column-by-column, or diagonal-by-diagonal.

# Memoization (top-down)

Use divide-and-conquer to fill-in the table.

- Return value if already computed.
- Recurse otherwise.
- Save solution in table before returning.

Pros:

- If some subproblem is irrelevant, memoization won't solve it.
- If you cannot figure out how to fill the table, divide-and-conquer will do it for you.

Cons:

- Recursive function calling overhead (stack frame).
- Sometimes miss tricks like we used in `FastDPCoinChange`.