The Folk Theorem

ISCI 330

April 3 and April 5, 2007

Infinitely Repeated Games

Definition

Given an infinite sequence of payoffs $r_1, r_2, \ldots$ for player i, the average reward of i is $\lim_{k\to\infty} \Sigma_{j=1}^{k} r_j/k$.

Definition

Given an infinite sequence of payoffs $r_1, r_2, \ldots$ for player i and a discount factor β with $0 \leq \beta \leq 1$, the future discounted rewards of i is $\sum_{j=1}^{\infty} \beta^j r_j$.

Nash Equilibria

- With an infinite number of equilibria, what can we say about Nash equilibria?
  - we won't be able to construct an induced normal form and then appeal to Nash's theorem to say that an equilibrium exists
  - Nash's theorem only applies to finite games
- Furthermore, with an infinite number of strategies, there could be an infinite number of pure-strategy equilibria!
- It turns out we can characterize a set of payoffs that are achievable under equilibrium, without having to enumerate the equilibria.

Definitions

- Consider any n-player game $G = (N, (A_i), (u_i))$ and any payoff vector $r = (r_1, r_2, \ldots, r_n)$.
- Let $v_i = \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_{-i}, s_i)$.
  - the amount of utility i can get when $-i$ play a minmax strategy against him

Definition

A payoff profile r is enforceable if $r_i \geq v_i$.

Definition

A payoff profile r is feasible if there exist rational, non-negative values α_a such that for all i, we can express r_i as $\sum_{a \in A} \alpha u_i(a)$, with $\sum_{a \in A} \alpha_a = 1$.

- a payoff profile is feasible if it is a convex, rational combination of the outcomes in G.

Folk Theorem

Theorem (Folk Theorem)

Consider any n-player game G and any payoff vector $(r_1, r_2, \ldots, r_n)$.

1. *If r is the payoff in any Nash equilibrium of the infinitely repeated G with average rewards, then for each player i, r_i is enforceable.*
2. *If r is both feasible and enforceable, then r is the payoff in some Nash equilibrium of the infinitely repeated G with average rewards.*

Folk Theorem (Part 1)

Payoff in Nash $\rightarrow$ enforceable

Part 1: Suppose r is not enforceable, i.e. $r_i < v_i$ for some i. Then consider a deviation of this player i to $b_i(s_{-i}(h))$ for any history h of the repeated game, where b_i is any best-response action in the stage game and $s_{-i}(h)$ is the equilibrium strategy of other players given the current history h. By definition of a minmax strategy, player i will receive a payoff of at least v_i in every stage game if he adopts this strategy, and so i's average reward is also at least v_i. Thus i cannot receive the payoff $r_i < v_i$ in any Nash equilibrium.

Folk Theorem (Part 2)

Feasible and enforceable $\longrightarrow$ Nash

Part 2: Since r is a feasible enforceable payoff profile, we can write it as $r_i = \sum_{a \in A}(\frac{\beta_a}{\gamma})u_i(a)$, where β_a and γ are non-negative integers. (Recall that α_a were required to be rational. So we can take γ to be their common denominator.) Since the combination was convex, we have $\gamma = \sum_{a \in A} \beta_a$.
We're going to construct a strategy profile that will cycle through all outcomes $a \in A$ of G with cycles of length γ, each cycle repeating action a exactly β_a times. Let (a^t) be such a sequence of outcomes. Let's define a strategy s_i of player i to be a trigger version of playing (a^t): if nobody deviates, then s_i plays a_i^t in period t. However, if there was a period t' in which some player $j \neq i$ deviated, then s_i will play $(p_{-j})_i$, where (p_{-j}) is a solution to the minimization problem in the definition of v_j.

Folk Theorem (Part 2)

Feasible and enforceable $\rightarrow$ Nash

First observe that if everybody plays according to s_i, then, by construction, player i receives average payoff of r_i (look at averages over periods of length γ). Second, this strategy profile is a Nash equilibrium. Suppose everybody plays according to s_i, and player j deviates at some point. Then, forever after, player j will receive his $\min\max$ payoff $v_j \leq r_j$, rendering the deviation unprofitable.