Compact Representations of Coalitional Games

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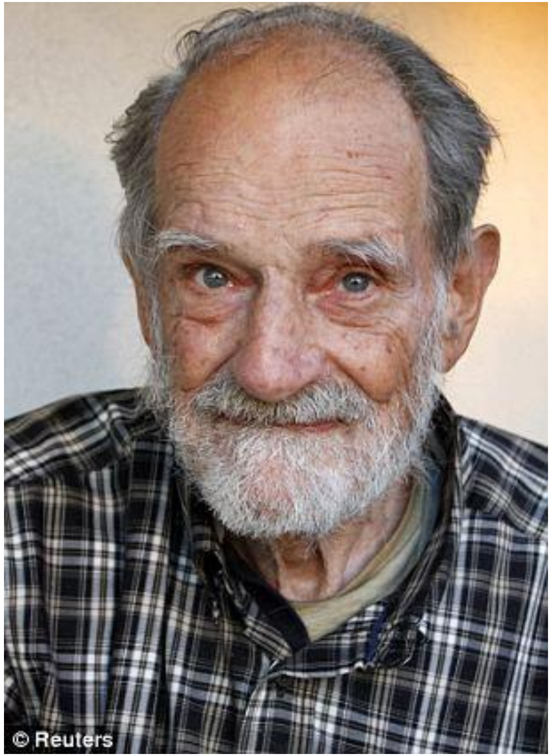
CPSC 532L

The Shapley Value is Useful 😀

- A fair way to divide the payoff of a coalition among its players
- Has many ML applications - stay tuned for the next talk



Lloyd S. Shapley

Computing the Shapley Value is Tedious 😔

We have to calculate the marginal contributions of each player, averaged over all possible orderings of how the coalition can be formed

- If there are *n* players, we must average over *n!* orderings of the players
- Computing the Shapley value becomes impractical very fast

Q: Is there a more efficient way to compute the shapley value?

Problem with the Naive Representation 😕

Even before that, how do we input the coalitional game into such programs?

Michael: *How can computing the core be polynomial if the game has an exponential number of subsets of N?*

Kevin: *It is polynomial to the size of the input, if your input size is exponential then too bad.*

Rest of the class: **audible laughter**

Motivations for Compact Representations

- We want more **efficient** ways to compute the solution concepts
- We want more **compact** ways to represent coalitional games

Presentation Outline

Motivation ✔

Overview of Compact Representations

I. Weighted Graph

II. Marginal Contribution Nets

Related Works

Takeaways

Compact Representation 👏

Representations that require at most polynomial space in the number of players

Limitations

- Tradeoff between the compactness of the representation, and the complexity of the associated computational problems
- Representations may not be able to cover all coalitional games

Weighted Graphs

Weighted Graphs

Proposed by Deng and Papadimitriou (1994)

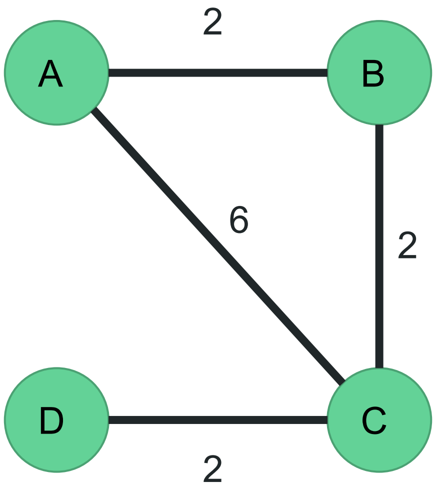
Idea: Represent the coalitional game as an undirected, weighted graph

- Vertices ➔ Players
- Edges ➔ Some integer
- Value of a coalition ➔ The weight of its induced subgraph

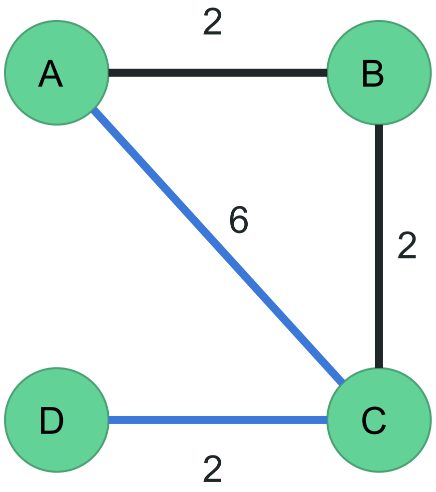
Games that are represented this way are called **induced subgraph games**

Weighted Graphs Representation

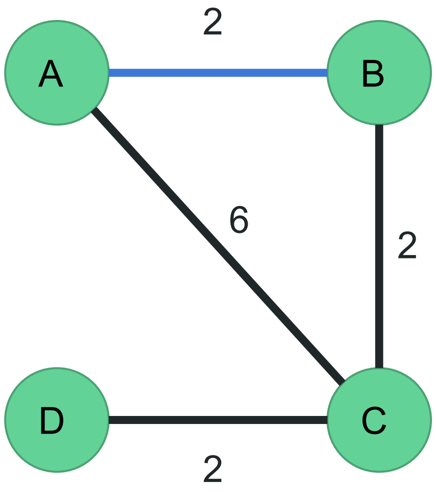
Consider a game with players $N = \{A, B, C, D\}$



Weighted graph representation



v({A, C, D}) = 2 + 6 = 8



v({A, B, D}) = 2

Weighted Graphs Shapley Value Computation

1. Consider every edge in the graph to be a separate game
2. Compute the Shapley value of a player in each edge game and sum them up

   Why does this work?   (Hint: one of Shapley's axioms)

Weighted Graphs Shapley Value Computation

1. Consider every edge in the graph to be a separate game
2. Compute the Shapley value of a player in each edge game and sum them up (Shapley's axiom: Additivity)
   a. Players gets a value of 0 for an edge they are not connected to
   b. Players gets half the weight of an edge they are connected to (Shapley's axiom: Symmetry)

Weighted Graphs Properties

Compact ✔ for a game with n players, we only need $O(n^2)$ space

Not Complete ❌ there are games that the weighted graph can't represent (e.g. a majority voting game)

Computing the **Shapley value**: Polynomial

Marginal-Contribution Nets

Marginal-Contribution Nets

Proposed by Ieong and Shoham (2005)

Idea: Use a set of rules to describe the marginal contributions of the players

- Rules are in the form

$$\textit{Pattern} \mapsto \textit{value}$$

- Patterns ➔ boolean condition over the set of players
- Value of a coalition ➔ sum of the values of all rules that apply to the coalition

MC-Nets Representation

Consider a game with players $N = \{A, B, C, D\}$

$\{A\} \mapsto 1$

$\{A \wedge B\} \mapsto 2$

$\{B \wedge D\} \mapsto 7$

$\{A \wedge D \wedge C\} \mapsto 6$

MC-Net Representation in the basic form (only conjunctions)

$v(\{A, B\}) = 1 + 2 = 3$

$v(\{A, B, D\}) = 1 + 2 + 7 = 10$

$v(\{B, C, D\}) = 7$

MC-Nets (Basic Form) Shapley Value Computation

1. Consider each rule as a separate game
2. Compute the Shapley value of a player in each rule and add them together (Shapley's axiom: additivity)
    a. For each rule a player belongs to, the player gets the value of that rule divided by the number of players in the rule (Shapley's axiom: symmetry)

MC-Nets are a generalization of Weighted Graphs

MC-Nets Properties

Compact ✔ for a game with *m* subgames, where the largest subgame has *n* players, it takes $O(m2^n)$ space

Complete ✔* one rule for every possible coalition

- Trade off between representational power and computation efficiency

Computing the **Shapley value**: Linear in the basic form

Related Work to Consider

Our discussion has been focused on the Shapley value

- What about the core?

The methods described in this presentation rely on the coalitional game to have certain properties

- What about more general methods? (e.g. *read-once MC-nets*)

Takeaways 👋

We use compact representations to:

- Reduce the space required to represent coalitional games
- Improve computational complexity for solution concepts

Trade off of compact representations

- □ Representational power/Compactness ↔ □ Computation efficiency

References

X. Deng and C. Papadimitriou, “On the Complexity of Cooperative Solution Concepts,” Mathematics of Operations Research, vol. 19, no. 2, 1994, pp. 257–266.

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