CPSC 532L: Models of Strategic Behaviour **Spring 2023**

Assignment #11

Due: April 18$^{\text{th}}$, 2023, 1:00pm

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TA: Narun Raman

1 Auctions

Problem 1.1. [**5 points**] Find the optimal reserve price in the first-price and the second-price auction if the valuation of each bidder is distributed exponentially with mean $\frac{1}{\mu}$.

Problem 1.2. [**10 points**] Consider two goods (A and B) and three bidders. Michael has valuation 1 for A and B together (i.e., $v_M(AB) = 1$) and 0 otherwise. Shruthi has valuation 1 for A (i.e., $v_s(AB) = v_s(A) = 1$) and 0 otherwise. Jackie has valuation 1 for B and 0 otherwise. Compute the VCG allocation and payments when only Michael and Shruthi are present. Do the same when all three are present. Can adding an extra bidder ever decrease the revenue of the Vickrey (single-item) auction? Give a brief explanation.

2 Algorithmic Mechanism Design

Problem 2.1. [**12 points**] The *knapsack problem* is a famous NP-hard problem in combinatorial optimization. The problem is stated as follows: There is a knapsack, which can hold a maximum weight of W. There are n items; each item i has weight $w_i \leq W$ and value v_i. The goal is to find some subset S of items of maximal total value with total weight no more than W. Written as an integer program,

$$\max \sum_{i=1}^{n} x_i v_i,$$

subject to

$$\sum_{i=1}^{n} x_i w_i \leq W, x \in \{0, 1\}.$$

We can frame this problem as a mechanism design problem as follows. Each bidder has an item that they would like to put in the knapsack. Each item has a public parameter w_i and a private value v_i. An auction will take place, after which some set S of bidders will place their items (of total weight less than W) in the knapsack and pay some amount of money to the auctioneer. Since the problem is NP-hard, we cannot hope to find a polynomial-time welfare-maximizing solution. Instead, we will produce a polynomial-time, DSIC mechanism that is a $\frac{1}{2}$-approximation (Meaning that for any set possible set of valuations and weights, we always achieve at least 50% of the optimal welfare.) of the optimal welfare. We propose the following greedy allocation scheme: Sort the bidders' items by their ratios v_i/w_i and allocate items in that order until there is no room left in the knapsack.

(a) [**3 points**] Show that this allocation scheme is not a $\frac{1}{2}$-approximation by producing an example where it fails to achieve 50% of the optimal welfare.

(b) [**6 points**] Prayus proposes a small improvement to the greedy allocation scheme. His improved allocation scheme compares 1) the welfare achieved by the greedy allocation scheme to 2) the welfare achieved by simply putting the single item of highest value into the knapsack and stopping. Prayus' allocation scheme then uses whichever of the two approaches maximizes welfare. Now, we would like to use this allocation scheme to create a mechanism that satisfies individual rationality and incentive compatibility. To do this, we will use Myerson's theorem. First, argue that Prayus' allocation scheme is monotone.

(c) [**3 points**] Use Myerson's payment formula to produce payments such that the resulting mechanism is DSIC.

3 Beyond Optimal Auctions

Problem 3.1. [**7 points**] In this problem, we will work through a concrete example of the Bulow-Klemperer theorem. Let us consider an auction where each buyer's valuation is drawn independently at random from the uniform distribution over $[0, a]$.

$$f(v) = \frac{1}{a} \qquad F(v) = \frac{v}{a}.$$

The k-th order statistic (expected value of the k-th largest sample) for this distribution is

$$\mathbb{E}[X_{(k)}] = a \cdot \frac{n-k+1}{n+1}.$$

We will show that a second-price sealed-bid auction (a.k.a. Vickrey auction) on 2 players does indeed generate at least as much revenue as the optimal auction (a.k.a. Meyerson auction) on 1 player. ***Please leave all solutions in terms of*** a

(a) [**1 point**] Derive an expression for the virtual value function $\phi(v)$ and find the optimal reserve price

(b) [**1 point**] Find the expected revenue of a second-price sealed-bid auction on 1 buyer

(c) [**2 points**] Find the expected revenue of the optimal auction on 1 buyer

(d) [**2 points**] Find the expected revenue of a second-price sealed-bid auction with a **random reserve price** drawn from f

(e) [**1 point**] Compare the expected revenue generated by the auctions in parts (b), (d), and (c) with (e), a second-price sealed-bid no-reserve-price auction on 2 buyers. Which ones are efficient?

Problem 3.2. [**3 points**] In this problem, we analyze the Single Sample mechanism, which,

- Picks a random bidder i with valuation v_i
- Runs a second-price, sealed-bid auction on the other bidders with reserve price v_i

Show that the Single Sample mechanism is:

(a) [**2 points**] truthful, regardless of who is chosen to be the reserve bidder

(b) [**1 point**] not efficient

4 Fun Bonus Questions: Earn Up To Ten Points

Knapsack Auctions

Problem 4.1. [**5 points**] Show that Prayus' allocation scheme is a $\frac{1}{2}$-approximation of optimal welfare.

First- and Second-Price Auctions

Problem 4.2. [10 points] We compare the revenue achieved by first- and second-price auctions for a single item. Analyzing what happens in a first-price auction is not trivial; we simplify things and assume that each valuation v_i is drawn i.i.d. from a known prior distribution F. A strategy of a bidder i in a first-price auction is some predetermined formula for (under)bidding: formally, a function $b_i(v_i)$ that maps its valuation v_i to a bid $b_i(v_i)$. You should conceptually think of this strategy (i.e., this function) as being announced to all of the other bidders in advance; but of course, the other bidders do not know the actual value of v_i (and hence do not know the corresponding bid $b_i(v_i)$). We will call such a family $b_1(\cdot), \dots, b_n(\cdot)$ of bidding functions a (Bayes-Nash) equilibrium if for every bidder i and every valuation v_i, the bid $b_i(v_i)$ maximizes i's expected payoff, where the expectation is with respect to the random draws of the other bidders' valuations (which, via their bidding functions, induce a distribution over their bids).

(a) [**5 points**] Suppose each valuation is an independent draw from $U(0, 1)$. Prove that one equilibrium is given by setting $b_i(v_i) = v_i(n-1)/n$ for every i and v_i.

(b) [**5 points**] Prove that the expected revenue of the seller at this equilibrium of the first-price auction is exactly the expected revenue of the seller with truthful bidding in a Vickrey auction (where in both cases the expectation is over the valuation draws).

Winner's Curse

Problem 4.3. [10 points] As the end of the semester approaches, Narun is sick and tired of Kevin getting all the fun in running auctions in class. As a result he decides to run an auction, outside of class hours! He selects two students that he knows will participate: Jenny and Ruiyu. He asks them to each privately check how much money is in their wallets (w_1, w_2, respectively). He announces that he will auction a prize equal to the combined contents of the wallets to Jenny and Ruiyu using a standard ascending (English) auction. That is, the price goes up continuously until Jenny (Ruiyu) quits, say at price p. At which point, Ruiyu (Jenny) pays p and receives $w_1 + w_2$.

(a) [**3 points**] How should Jenny and Ruiyu bid? Can you find an equilibrium in which the strategies are symmetric? If not, explain why.

(b) [**4 points**] Suppose now that if Jenny wins the auction she earns a small bonus prize of \$1. (Ruiyu receives no bonus for winning.) How does a small bonus of \$1 affect the bidding?

(c) [**3 points**] Does your analysis above make you reconsider the auction that was run? If so, what auction might you want to run instead and why? If not, explain why.