Imperfect Information Extensive Form Games
Game Theoretic Analysis

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**Perfect Information Extensive Form Game**

**Definition**

A **finite perfect information game in extensive form** is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u)$, where

- $N$ is a set of $n$ players
- $A$ is a single set of actions
- $H$ is a set of nonterminal choice nodes
- $Z$ is a set of terminal nodes (disjoint from $H$)
- $\chi : H \rightarrow 2^A$ is the action function
- $\rho : H \rightarrow N$ is the player function
- $\sigma : H \times A \rightarrow H \cup Z$ is the successor function
- $u = (u_1, \ldots, u_n)$ is a profile of utility functions $u_i : Z \rightarrow \mathbb{R}$ for each player $i$
Recap: Pure Strategies

Definition

Let $G = (N, A, H, Z, \chi, \rho, \sigma, u)$ be a perfect information game in extensive form. Then the pure strategies for player $i$ consist of the cross product of actions available to $i$ at each of their choice nodes:

$$\prod_{h \in H | \rho(h) = o} \chi(h).$$

Note that a pure strategy associates an action with every choice node, even those that will never be reached.
**Recap: Induced Normal Form**

- Any pair of pure strategies uniquely identifies a **terminal node**, which identifies a **utility** for each agent.
- We have now defined a set of **agents**, **pure strategies**, and **utility functions**.
- Any perfect-information extensive form game defines a corresponding **induced normal form game**.

![Game Tree Diagram]

### Pure Strategies Example

<table>
<thead>
<tr>
<th>Action</th>
<th>C, E</th>
<th>C, F</th>
<th>D, E</th>
<th>D, F</th>
</tr>
</thead>
<tbody>
<tr>
<td>A, G</td>
<td>3,8</td>
<td>3,8</td>
<td>8,3</td>
<td>8,3</td>
</tr>
<tr>
<td>A, H</td>
<td>3,8</td>
<td>3,8</td>
<td>8,3</td>
<td>8,3</td>
</tr>
<tr>
<td>B, G</td>
<td>5,5</td>
<td>2,10</td>
<td>5,5</td>
<td>2,10</td>
</tr>
<tr>
<td>B, H</td>
<td>5,5</td>
<td>1,0</td>
<td>5,5</td>
<td>1,0</td>
</tr>
</tbody>
</table>
Recap: Backward Induction

- **Backward induction** is a straightforward algorithm that is guaranteed to compute a **pure strategy, subgame perfect equilibrium**

- **Idea:** Replace subgames with their equilibrium values

\[
\begin{align*}
A, C, F, G: & (3, 8) \\
A, C, F: & (3, 8) \\
C: & (3, 8) \\
B: & (5, 5) \\
E: & (2, 10) \\
F: & (2, 10) \\
G: & (2, 10) \\
H: & (1, 0)
\end{align*}
\]

\[
(A, G), (C, F)
\]
Lecture Overview

Imperfect Information Games

Behavioural vs. Mixed Strategies

Perfect vs. Imperfect Recall
Imperfect Information, informally

- **Perfect information** extensive form games model sequential actions that are observed by all players.
Imperfect Information, informally

- **Perfect information** extensive form games model sequential actions that are observed by all players.
- But many games involve **hidden actions**
  - Cribbage, poker, Scrabble
  - Sometimes the actions of the players are hidden, sometimes “Nature’s” actions are hidden, sometimes both.
Imperfect Information, informally

- **Perfect information** extensive form games model sequential actions that are observed by all players.
- But many games involve **hidden actions**
  - Cribbage, poker, Scrabble
  - Sometimes the actions of the players are hidden, sometimes “Nature’s” actions are hidden, sometimes both.
- **Imperfect information extensive form games** are a model of games with sequential actions, some of which may be hidden.
Imperfect Information Extensive Form Game

Definition

An imperfect information game in extensive form is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u, \mathcal{I})$, where

- $(N, A, H, Z, \chi, \rho, \sigma, u)$ is a perfect information extensive form game, and
- $\mathcal{I} = (I_1, \ldots, I_n)$, where for each $i \in N$, $\mathcal{I}_i = (I_{i,1}, \ldots, I_{i,k_i})$ is an equivalence relation on (i.e., partition of) $\{h \in H \mid \rho(h) = i\}$, and
- For every $h, h'$ such that $h \in I$ and $h' \in I$ for some $i \in N$ and $I \in \mathcal{I}_i$, $\rho(h) = \rho(h') = i$ and $\chi(h) = \chi(h')$. 

Game Theoretic Analysis: Imperfect Information EFGs: Leyton-Brown & Wright (8)
Imperfect Information EFG Example

- The elements of the partition are often called **information sets**
- Players cannot distinguish which **history** they are in within an information set
Imperfect Information EFG Example

- The elements of the partition are often called **information sets**
- Players cannot distinguish which **history** they are in within an information set
- **Question**: What are the information sets for each player in this game?
Pure Strategies

**Question:** What are **pure strategies** in an imperfect information extensive form game?
**Pure Strategies**

**Question:** What are pure strategies in an imperfect information extensive form game?

**Definition**

Let $G = (N, A, H, Z, \chi, \rho, \sigma, u, \mathcal{I})$ be an imperfect information game in extensive form. Then the set of pure strategies of player $i$ is the cross product of actions available to player $i$ at each of their information sets, i.e.,

$$\prod_{I \in \mathcal{I}_i} \chi(I),$$

where $\chi(I) = \chi(h)$ for an arbitrary $h \in I$. 
**Pure Strategies**

**Question:** What are pure strategies in an imperfect information extensive form game?

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**Questions**

In an imperfect information EFG:

1. What are the mixed strategies?
Question: What are pure strategies in an imperfect information extensive form game?

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Questions

In an imperfect information EFG:

1. What are the mixed strategies?
2. What is a best response?
**Pure Strategies**

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Let $G = (N, A, H, Z, \chi, \rho, \sigma, u, \mathcal{I})$ be an imperfect information game in extensive form. Then the set of **pure strategies of player** $i$ is the cross product of actions available to player $i$ at each of their **information sets**, i.e.,

$$
\prod_{I \in \mathcal{I}_i} \chi(I),
$$

where $\chi(I) = \chi(h)$ for an arbitrary $h \in I$.

**Questions**

In an imperfect information EFG:

1. What are the mixed strategies?
2. What is a best response?
3. What is a Nash equilibrium?

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*Game Theoretic Analysis: Imperfect Information EFGs: Leyton-Brown & Wright (10)*
Induced Normal Form

- Any pair of pure strategies uniquely identifies a terminal node, which identifies a utility for each agent
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- We have now defined a set of agents, pure strategies, and utility functions
**Induced Normal Form**

- Any pair of pure strategies uniquely identifies a terminal node, which identifies a utility for each agent.
- We have now defined a set of agents, pure strategies, and utility functions.
- Any imperfect information extensive form game defines a corresponding *induced normal form game*.
Induced Normal Form

Any pair of pure strategies uniquely identifies a terminal node, which identifies a utility for each agent.

We have now defined a set of agents, pure strategies, and utility functions.

Any imperfect information extensive form game defines a corresponding induced normal form game.

Question

Can you represent an arbitrary perfect information EFG as an imperfect information EFG?
Normal to Extensive Form

- Unlike perfect information EFGs, we can also represent any normal form game as an imperfect information EFG.
- Players can play in any order (why?)
Normal to Extensive Form

- Unlike perfect information EFGs, we can also represent any normal form game as an imperfect information EFG.
- Players can play in any order (why?)
- Question: What happens if we run this NFG→EFG translation on the induced normal form of an arbitrary extensive form game?
Lecture Overview

Imperfect Information Games

Behavioural vs. Mixed Strategies

Perfect vs. Imperfect Recall
Definition

A **mixed strategy** $s_i$ (in an imperfect information EFG) is any distribution over an agent’s pure strategies:

$$s_i \in \Delta(A^I_i)$$
Beckhavioural vs. Mixed Strategies

Definition

A **mixed strategy** \( s_i \) (in an imperfect information EFG) is any distribution over an agent’s pure strategies:

\[
s_i \in \Delta(A^I_i)
\]

Definition

A **behavioral strategy** \( b_i \) is a mapping from an agent’s information sets to a distribution over the actions at that information set, which is sampled independently each time the agent arrives at the information set:

\[
b_i \in [\Delta(\chi(I))]_{I \in I_i}
\]
Behavioural vs. Mixed Example

- Behavioural strategy: \([\{0.6 : A, 0.4 : B\}, \{0.6 : G, 0.4 : H\}]\)
- Mixed strategy: \([0.6 : (A, G), 0.4 : (B, H)]\)
Recap

Imperfect Information Games

Behavioural vs. Mixed Strategies

Perfect vs. Imperfect Recall

Summary

**Behavioural vs. Mixed Example**

- **Behavioural strategy:** \([.6 : A, .4 : B], [.6 : G, .4 : H]\)
- **Mixed strategy:** \([.6 : (A, G), .4 : (B, H)]\)
- **Question:** Are these strategies equivalent? *(why?)*

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**Pure Strategies Example**

**Games with Sequential Actions: Reasoning and Computing with the Extensive Form**

<table>
<thead>
<tr>
<th>Action 1</th>
<th>Action 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A</strong></td>
<td><strong>B</strong></td>
</tr>
<tr>
<td><strong>C</strong></td>
<td><strong>D</strong></td>
</tr>
<tr>
<td><strong>E</strong></td>
<td><strong>F</strong></td>
</tr>
<tr>
<td><strong>G</strong></td>
<td><strong>H</strong></td>
</tr>
</tbody>
</table>

- **Payoffs:**
  - \((3, 8)\)
  - \((8, 3)\)
  - \((5, 5)\)
  - \((2, 10)\)
  - \((1, 0)\)

- **Figure 5.2:** A perfect-information game in extensive form.
- **Figure 5.3:** The game from Figure 5.2 in normal form.

A perfect-information game can be converted to an equivalent normal-form game. For example, the perfect-information game of Figure 5.2 can be converted into the normal form image of the game, shown in Figure 5.3. Clearly, the strategy spaces of the two games are the same, as are the pure-strategy Nash equilibria. (Indeed, both the mixed strategies and the mixed-strategy Nash equilibria of the two games are also the same; however, we defer further discussion of mixed strategies until we consider imperfect-information games in Section 5.2.)

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Game Theoretic Analysis: Imperfect Information EFGs: Leyton-Brown & Wright (15)
**Behavioural vs. Mixed Example**

- **Behavioural strategy:** \([.6 : A, .4 : B], [.6 : G, .4 : H]\)
- **Mixed strategy:** \([.6 : (A, G), .4 : (B, H)]\)
- **Question:** Are these strategies equivalent? (why?)
- **Question:** Can you construct a mixed strategy that is equivalent to the behavioral strategy above?

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Game Theoretic Analysis: Imperfect Information EFGs: Leyton-Brown & Wright (15)
Behavioural vs. Mixed Example

- Behavioural strategy: \([.6:A, .4:B], [.6:G, .4:H]\)
- Mixed strategy: \([.6:(A,G), .4:(B,H)]\)
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- **Question:** Can you construct a behavioural strategy that is equivalent to the mixed strategy above?
Lecture Overview

Imperfect Information Games

Behavioural vs. Mixed Strategies

Perfect vs. Imperfect Recall
Perfect Recall

Definition

Player \(i\) has **perfect recall** in an imperfect information game \(G\) if for any two choice nodes \(h, h' \in I\) that are in the same information set \(I \in \mathcal{I}_i\) for player \(i\), for any path \(h_0, a_0, h_1, a_1, \ldots, h_n, h\) from the root of the game to \(h\), and for any path \(h'_0, a'_0, h'_1, a'_1, \ldots, h'_m, h'\) from the root of the game to \(h'\), it must be the case that:

1. \(n = m\), and
2. for all \(0 \leq j \leq n\), if \(\rho(h_j) = i\), then there exists an information set \(I' \in \mathcal{I}_i\) such that \(h_j, h'_j \in I'\) (i.e., \(h_j, h'_j\) are in the same information set), and
3. for all \(0 \leq j \leq n\), if \(\rho(h_j) = i\), then \(a_j = a'_j\).

\(G\) is a game of perfect recall if every player has perfect recall in \(G\).
Perfect Recall

**Definition**

Player $i$ has **perfect recall** in an imperfect information game $G$ if for any two choice nodes $h, h' \in I$ that are in the same information set $I \in \mathcal{I}_i$ for player $i$, for any path $h_0, a_0, h_1, a_1, \ldots, h_n, h$ from the root of the game to $h$, and for any path $h'_0, a'_0, h'_1, a'_1, \ldots, h'_m, h'$ from the root of the game to $h'$, it must be the case that:

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3. for all $0 \leq j \leq n$, if $\rho(h_j) = i$, then $a_j = a'_j$. 

Game Theoretic Analysis: Imperfect Information EFGs: Leyton-Brown & Wright (17)
Perfect Recall

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Player $i$ has **perfect recall** in an imperfect information game $G$ if for any two choice nodes $h, h' \in I$ that are in the same information set $I \in \mathcal{I}_i$ for player $i$, for any path $h_0, a_0, h_1, a_1, \ldots, h_n, h$ from the root of the game to $h$, and for any path $h'_0, a'_0, h'_1, a'_1, \ldots, h'_m, h'$ from the root of the game to $h'$, it must be the case that:

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3. for all $0 \leq j \leq n$, if $\rho(h_j) = i$, then $a_j = a'_j$.

$G$ is a **game of perfect recall** if every player has perfect recall in $G$. 

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Game Theoretic Analysis: Imperfect Information EFGs: Leyton-Brown & Wright (17)
Perfect Recall Examples

Question
Which of the above games is a game of perfect recall?
Imperfect Recall Example

![Diagram of an imperfect recall game]

Figure 5.12: A game with imperfect recall

Question: Can you construct a mixed strategy equilibrium in this game?

• Can you construct a behavioural strategy equivalent to the mixed strategy \([0.5:L, 0.5:R]\)?

Player 1 doesn't remember whether they played \(L\) or \(R\) in the previous round.

There is, however, a broad class of imperfect-information games in which the expressive power of mixed and behavioural strategies coincide.

Definition 5.2.3 (Perfect recall) is a game of perfect recall if every player has perfect recall of games of perfect recall.

Clearly, every perfect-information game is a game of perfect recall.

Multiagent Systems

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Imperfect Recall Example

- Player 1 **doesn’t remember** whether they played $L$ before or not. In this case, that is because they visit the **same information set multiple times**.
Imperfect Recall Example

- Player 1 *doesn’t remember* whether they played $L$ before or not. In this case, that is because they visit the *same information set multiple times*. 

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Game Theoretic Analysis: Imperfect Information EFGs: Leyton-Brown & Wright (19)
Imperfect Recall Example

- Player 1 **doesn’t remember** whether they played $L$ before or not. In this case, that is because they visit the **same information set multiple times**.
- **Question:** Can you construct a **mixed strategy** equivalent to the behavioral strategy $[.5 : L, .5 : R]$ in this game?

![Game Diagram](image-url)
Imperfect Recall Example

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- **Question:** Can you construct a **behavioural strategy** equivalent to the mixed strategy \([.5: (L), .5: (R)]\) in this game?
Imperfect Recall Example

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- **Question:** What is the mixed strategy equilibrium of this game?
Imperfect Recall Example

- Player 1 **doesn’t remember** whether they played L before or not. In this case, that is because they visit the **same information set multiple times**.

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- **Question:** Can you construct a **behavioural strategy** equivalent to the mixed strategy [.5:(L), .5:(R)] in this game?

- **Question:** What is the **mixed strategy equilibrium** of this game?

- **Question:** What is the **equilibrium in behavioural strategies** of this game?
Question: When is it useful to model a scenario as a game of imperfect recall?
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1. When the actual agents may forget previous history
   - Including cases where the agent strategies are really executed by proxies
Imperfect Recall Applications

Question: When is it *useful* to model a scenario as a game of imperfect recall?

1. When the **actual agents** may forget previous history
   - Including cases where the agent strategies are really executed by proxies
2. As an **approximation technique**
   - E.g., poker: The exact exact cards that have been played to this point may not matter as much as some coarse grouping of which cards have been played
   - Grouping the cards into equivalence classes is a **lossy** approximation
Kuhn’s Theorem

**Theorem** [Kuhn, 1953]

In a game of perfect recall, any mixed strategy of a given agent can be replaced by an equivalent behavioural strategy, and any behavioural strategy can be replaced by an equivalent mixed strategy.
Kuhn’s Theorem

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In a game of perfect recall, any mixed strategy of a given agent can be replaced by an equivalent behavioural strategy, and any behavioural strategy can be replaced by an equivalent mixed strategy.

- Two strategies are equivalent when they induce the same probabilities on outcomes, for any fixed strategy profile of the other agents.
Kuhn’s Theorem

**Theorem** [Kuhn, 1953]

In a game of perfect recall, any *mixed strategy* of a given agent can be replaced by an *equivalent behavioural strategy*, and any *behavioural strategy* can be replaced by an *equivalent mixed strategy*.

- Two strategies are *equivalent* when they induce the same probabilities on outcomes, for any fixed strategy profile of the other agents.

**Corollary**

Restricting attention to behavioral strategies does not change the set of Nash equilibria in a game of perfect recall. (*why*)
Summary

- **Imperfect information extensive form games** are a model of games with sequential actions, some of which may be **hidden**
  - Histories are partitioned into **information sets**
  - Players **cannot distinguish** between histories in the same information set
- **Pure strategies** map each information set to an action
  - Mixed strategies are distributions over **pure strategies**
  - Behavioural strategies map each information set to a distribution over **actions**
  - In games of **perfect recall**, mixed strategies and behavioral strategies are interchangeable