Extensive Form Games

Game Theoretic Analysis

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Recap: Best Response & Nash Equilibrium

**Definition**

The set of \( i \)'s best responses to a strategy profile \( s_{-i} \in S_{-i} \) is

\[
BR_i(s_{-i}) = \{ a_i^* \in A_i \mid u_i(a_i^*, s_{-i}) \geq u_i(a_i, s_{-i}) \quad \forall a_i \in A_i \}
\]

**Definition**

A strategy profile \( s \) is a **Nash equilibrium** iff

\[
\forall i \in N, s'_i \in S_i : u_i(s) \geq u_i(s'_i, s_{-i})
\]

Equivalently,

\[
\forall i \in N, a_i \in A_i : s_i(a_i) > 0 \iff a_i \in BR_i(s_{-i}).
\]

When at least one \( s_i \) is mixed, \( s \) is a **mixed strategy Nash equilibrium**
Recap: Best Response & Nash Equilibrium

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When at least one \( s_i \) is mixed, \( s \) is a \textbf{mixed strategy Nash equilibrium}
Recap: Rationalizability

A rationalizable strategy is one which is a best response to some belief about the other agents

- that also assumes opponent is playing some rationalizable strategy
- the beliefs need not be consistent with each other

In two-player games, rationalizable strategies are exactly those that survive *iterated removal of strictly dominated strategies*. 
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Example: Traveller’s Dilemma

- 300 is weakly dominated by 299
- But it is strictly dominated by a mixed strategy over the actions 180–299.
- So 300 does not survive iterated removal of strictly dominated strategies
- In the game with 300 removed, 299 is weakly dominated by 298
- ...but strictly dominated by a mixed strategy over 180–298
Lecture Overview

Extensive Form Games

Nash equilibrium

Subgame Perfect Equilibrium
Extensive Form Games

- Normal form games don’t have any notion of *sequence*: all actions happen **simultaneously**
- The **extensive form** is a game representation that explicitly includes temporal structure (i.e., a game tree)
There are two kinds of extensive form game:

1. **Perfect information:** Every agent **sees all actions** of the other players (including any special “Chance” player)
   - e.g., Chess, Checkers, Backgammon, Pandemic
   - This lecture!
There are two kinds of extensive form game:

1. **Perfect information**: Every agent sees all actions of the other players (including any special “Chance” player)
   - e.g., Chess, Checkers, Backgammon, Pandemic
   - This lecture!

2. **Imperfect information**: Some actions are hidden
   - Players may not know exactly where they are in the tree
   - Different players may have different knowledge (about where they are in the tree)
   - E.g., Poker, Rummy, Scrabble
A finite perfect information game in extensive form is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u)$, where

- $N$ is a set of $n$ players
- $A$ is a single set of actions
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**Perfect Information Extensive Form Game**

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- \( \sigma : H \times A \to H \cup Z \) is the successor function
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- $\rho : H \to N$ is the player function
- $\sigma : H \times A \to H \cup Z$ is the successor function
- $u = (u_1, \ldots, u_n)$ is a profile of utility functions $u_i : Z \to \mathbb{R}$ for each player $i$
Fun Game: The Sharing Game

- Two siblings must decide how to share two $100 coins
- Sibling 1 suggests a division, then sibling 2 accepts or rejects
  - If rejected, nobody gets any coins
- Play against 2 other people, once per person, different role each time
Fun Game: The Sharing Game

- Two siblings must decide how to share two $100 coins
- Sibling 1 suggests a division, then sibling 2 accepts or rejects
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- Play against 2 other people, once per person, different role each time
- **Question:** Did you have a plan for every possible eventuality?
Lecture Overview

Extensive Form Games

Nash equilibrium

Subgame Perfect Equilibrium
Pure Strategies

Question
What are the pure strategies in an extensive form game?
**Pure Strategies**

**Question**

What are the **pure strategies** in an extensive form game?

**Definition**

Let $G = (N, A, H, Z, \chi, \rho, \sigma, u)$ be a perfect information game in extensive form. Then the **pure strategies** for player $i$ consist of the cross product of actions available to $i$ at each of their choice nodes:

$$
\prod_{h \in H | \rho(h) = o} \chi(h).
$$

Note that a pure strategy associates an action with **every** choice node, even those that will **never be reached**.
Question: What are the pure strategies for player 2?

Pure Strategies Example

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\{ (C, E), (C, F), (D, E), (D, F) \}
Pure Strategies Example

Question: What are the pure strategies for **player 2**?

\{ (C, E), (C, F), (D, E), (D, F) \}

Question: What are the pure strategies for **player 1**?
Question: What are the pure strategies for player 2?

\{(C, E), (C, F), (D, E), (D, F)\}

Question: What are the pure strategies for player 1?

\{(A, G), (A, H), (B, G), (B, H)\}
Question: What are the pure strategies for player 2?

\{(C, E), (C, F), (D, E), (D, F)\}

Question: What are the pure strategies for player 1?

\{(A, G), (A, H), (B, G), (B, H)\}

Note that there is always an action for the second node, even when it cannot be reached.
Induced Normal Form

- Any pair of pure strategies uniquely identifies a **terminal node**, which identifies a **utility** for each agent (**why?**)

```
A
 /   \
/     /
/       /
C       D
|       |
(3,8)   (8,3)

B
 /   \
/     /
/       /
E       F
|       |
(5,5)   (5,5)

G
 /   \
/     /
(2,10)   (1,0)

H
```

*Game Theoretic Analysis: Extensive Form Games: Leyton-Brown & Wright (12)*
Induced Normal Form

- Any pair of pure strategies uniquely identifies a **terminal node**, which identifies a **utility** for each agent (why?)
- We have now defined a set of agents, pure strategies, and utility functions
- Any perfect-information extensive form game defines a corresponding **induced normal form game**

![Game Tree](image)

<table>
<thead>
<tr>
<th></th>
<th>C, E</th>
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<tbody>
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Induced Normal Form

- Any pair of pure strategies uniquely identifies a terminal node, which identifies a utility for each agent. Why?
- We have now defined a set of agents, pure strategies, and utility functions.
- Any perfect-information extensive form game defines a corresponding induced normal form game.
- Question: Which representation is more compact?
We can also plug our new definition of pure strategy into our existing definitions for:

- Mixed strategy
- Best response
- Nash equilibrium (both pure strategy and mixed strategy)
We can also plug our new definition of pure strategy into our existing definitions for:

- Mixed strategy
- Best response
- Nash equilibrium (both pure strategy and mixed strategy)

**Question**

What is the definition of a mixed strategy in an extensive form game?
Theorem [Zermelo, 1913]

Every finite perfect-information game in extensive form has at least one pure strategy Nash equilibrium.
Pure Strategy Nash Equilibria

**Theorem** [Zermelo, 1913]
Every finite perfect-information game in extensive form has at least one pure strategy Nash equilibrium.

**Proof:** Solve by **backward induction**

- Starting from the bottom of the tree, no agent needs to randomize, because there is a deterministic best response.
- Replace those nodes with the resulting utility vector
- Repeat until an action is assigned for all choice nodes

(There might be multiple pure strategy Nash equilibria in cases where an agent has multiple best responses at a single choice node.)
Question: What are the pure-strategy Nash equilibria of this game?

Game Theoretic Analysis: Extensive Form Games: Leyton-Brown & Wright (15)
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Question: Do any of them seem implausible?
**Pure Strategy Nash Equilibria Example**

**Question:** What are the pure-strategy Nash equilibria of this game?

**Question:** Do any of them seem implausible?

![Extensive Form Game Diagram](image)

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Lecture Overview

Extensive Form Games

Nash equilibrium

Subgame Perfect Equilibrium
Subgame Perfection, informally

Some equilibria seem **less plausible** than others.

- \[\{(B, H), (C, E)\}\]: \(F\) has payoff 0 for player 2, because player 1 plays \(H\), so player 2’s best response is to play \(E\).
- But why would player 1 play \(H\) if they got to that choice node?
- The equilibrium relies on a “threat” from player 1 that is not **credible**.
- **Subgame perfect equilibria** are Nash equilibria that do not rely on non-credible threats.
Subgames

Definition
The subgame of \( G \) rooted at \( h \) is the restriction of \( G \) to the descendants of \( h \).

Definition
The subgames of \( G \) are the subgames of \( G \) rooted at \( h \) for every choice node \( h \in H \).

!?Game Theoretic Analysis: Extensive Form Games: Leyton-Brown & Wright (18)!
Subgames

Definition

The **subgame of** $G$ **rooted at** $h$ is the restriction of $G$ to the descendants of $h$.

Examples:

For every choice node $h \in H$, the subgames of $G$ are the subgames of $G$ rooted at $h$. For example, in Figure 5.3 there are 16 different outcomes, whereas in Figure 5.2 there are only 5; the payoff vector $(2,10)$ occurs only once in Figure 5.2 but four times in Figure 5.3.
Subgame Perfect Equilibrium

**Definition**

A strategy profile $s \in S$ is a **subgame perfect equilibrium** of $G$ iff, for every subgame $G'$ of $G$, the restriction of $s$ to $G'$ is a Nash equilibrium of $G'$.
Subgame Perfect Equilibrium

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Any equilibrium computed by backward induction will be subgame perfect (**Why?**)
Summary

• **Extensive form games** allow us to represent sequential action
  – Perfect information: when we see everything that happens
  – Imperfect information: different agents have different information
• **Pure strategies** for extensive form games map choice nodes to actions
  – Induced normal form is the normal form game with these pure strategies
  – Notions of mixed strategy, best response, etc. translate directly
• **Subgame perfect equilibria** are those which do not rely on non-credible threats
  – Can always find a subgame perfect equilibrium using backward induction
  – Furthermore, this equilibrium is guaranteed to be in pure strategies