# Game-Theoretic Analysis

## Alternate Solution Concepts

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# Lecture Overview

**Computing Mixed Nash Equilibria**

Fun Game

Maxmin and Minmax

Iterated Removal of Dominated Strategies

Rationalizability

Correlated Equilibrium

# Computing Mixed Nash Equilibria: Battle of the Sexes

| | B | F |
|---|---|---|
| B | $2, 1$ | $0, 0$ |
| F | $0, 0$ | $1, 2$ |

- It's hard in general to compute Nash equilibria, but it's easy when you can guess the **support**
- For BoS, let's look for an equilibrium where all actions are part of the support

# Computing Mixed Nash Equilibria: Battle of the Sexes

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- Let player 2 play $B$ with $p$, $F$ with $1 - p$.
- If player 1 best-responds with a mixed strategy, player 2 must make her indifferent between $F$ and $B$ (why?)

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$$u_1(B) = u_1(F)$$
$$2p + 0(1-p) = 0p + 1(1-p)$$
$$p = \frac{1}{3}$$

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  - Why is player 1 willing to randomize?

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- Likewise, player 1 must randomize to make player 2 indifferent.
  - Why is player 1 willing to randomize?
- Let player 1 play $B$ with $q$, $F$ with $1 - q$.

$$\begin{aligned} u_2(B) &= u_2(F) \\ q + 0(1-q) &= 0q + 2(1-q) \\ q &= \frac{2}{3} \end{aligned}$$

- Thus the strategies $(\frac{2}{3}, \frac{1}{3})$, $(\frac{1}{3}, \frac{2}{3})$ are a Nash equilibrium.

# Lecture Overview

# Fun Game!

| | $L$ | $R$ |
|---|---|---|
| $T$ | $80, 40$ | $40, 80$ |
| $B$ | $40, 80$ | $80, 40$ |

- In a breakout room, play each game once as each player.

# Fun Game!

| | $L$ | $R$ |
|---|---|---|
| $T$ | $320, 40$ | $40, 80$ |
| $B$ | $40, 80$ | $80, 40$ |

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# Fun Game!

| | $L$ | $R$ |
|---|---|---|
| $T$ | 44, 40 | 40, 80 |
| $B$ | 40, 80 | 80, 40 |

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- What happens when people play this game?
  - with payoff of 320, row player goes up essentially all the time
  - with payoff of 44, row player goes down essentially all the time

# Lecture Overview

# Maxmin Strategies

- Player $i$'s **maxmin strategy** is a strategy that maximizes $i$'s worst-case payoff, in the situation where all the other players (whom we denote $-i$) happen to play the strategies which cause the greatest harm to $i$.
- The **maxmin value** (or **safety level**) of the game for player $i$ is that minimum amount of payoff guaranteed by a maxmin strategy.

## Definition (Maxmin)

The **maxmin strategy** for player $i$ is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$, and the **maxmin value** for player $i$ is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.

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- Why would $i$ want to play a maxmin strategy?
  - a conservative agent maximizing worst-case payoff
  - a paranoid agent who believes everyone is out to get him

# Minmax Strategies

- Player $i$'s **minmax strategy** against player $-i$ in a 2-player game is a strategy that minimizes $-i$'s best-case payoff, and the **minmax value** for $i$ against $-i$ is payoff.
- Why would $i$ want to play a minmax strategy?

## Definition (Minmax, 2-player)

In a two-player game, the **minmax strategy** for player $i$ against player $-i$ is $\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$, and player $-i$'s **minmax value** is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$.

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- Why would $i$ want to play a minmax strategy?
  - to punish the other agent as much as possible

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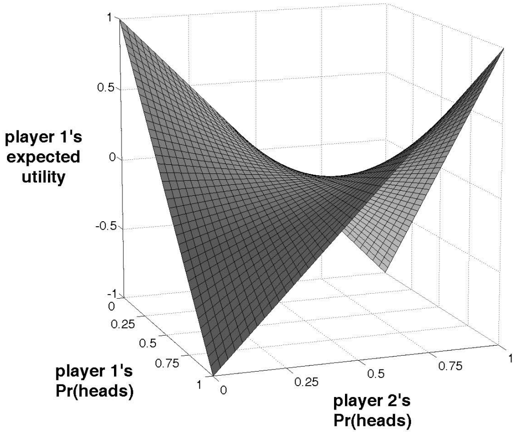
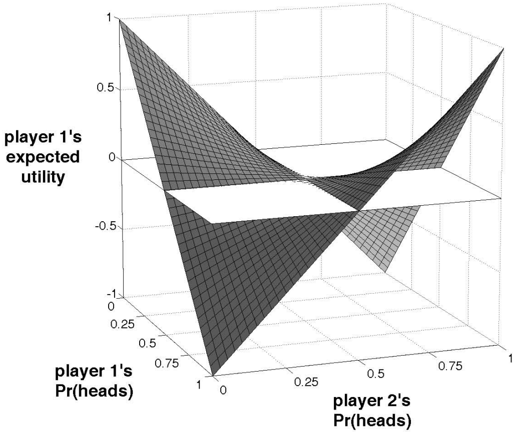
1. Each player's maxmin value is equal to his minmax value. By convention, the maxmin value for player 1 is called the **value of the game**.
2. For both players, the set of maxmin strategies coincides with the set of minmax strategies.
3. Any maxmin strategy profile (or, equivalently, minmax strategy profile) is a Nash equilibrium. Furthermore, these are all the Nash equilibria. Consequently, all Nash equilibria have the same payoff vector (namely, those in which player 1 gets the value of the game).

# Saddle Point: Matching Pennies



# Lecture Overview

# Traveler's Dilemma

*Two travelers purchase identical African masks while on a tropical vacation. Their luggage is lost on the return trip, and the airline asks them to make independent claims for compensation. In anticipation of excessive claims, the airline representative announces: "We know that the bags have identical contents, and we will entertain any claim between \$180 and \$300, but you will each be reimbursed at an amount that equals the minimum of the two claims submitted. If the two claims differ, we will also pay a reward $R$ to the person making the smaller claim and we will deduct a penalty $R$ from the reimbursement to the person making the larger claim."*

# Traveler's Dilemma

- Action: **choose an integer** between 180 and 300
- If both players **pick the same number**, they both get that amount as payoff
- If players **pick a different number**:
  - the low player gets his number $(L)$ plus some constant $R$
  - the high player gets $L - R$, $R = 5$.
- Play this game in a breakout room, if we have time. Do it once with $R = 5$, once with $R = 180$.

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## Traveler's Dilemma

- What is the equilibrium?
  - $(180, 180)$ is the only equilibrium, for all $R \geq 2$.
- What happens?
  - with $R = 5$ most people choose 295–300
  - with $R = 180$ most people choose 180

# Dominated strategies

- No equilibrium can involve a strictly dominated strategy
  - Thus we can remove it, and end up with a strategically equivalent game
  - This might allow us to remove another strategy that wasn't dominated before
  - Running this process to termination is called **iterated removal of dominated strategies**.

# Iterated Removal of Strictly Dominated Strategies

- This process **preserves all Nash equilibria**.
  - If there are multiple dominated strategies, the order of removal doesn't matter
- Thus, it can be used as a **preprocessing step** before computing an equilibrium
  - Some games are solvable using this technique
  - Example: Traveler's Dilemma!

# Lecture Overview

# Rationalizability

- Rather than ask what is irrational, ask what is a best response to **some** beliefs about the opponent
  - assumes opponent is rational
  - assumes opponent knows that you and the others are rational
  - ...
- Examples
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  - is $heads$ rationalizable in matching pennies?
  - is $cooperate$ rationalizable in prisoner's dilemma?
- Will there always exist a rationalizable strategy?
  - Yes, equilibrium strategies are always rationalizable.
- Furthermore, in two-player games, rationalizable $\Leftrightarrow$ survives iterated removal of strictly dominated strategies.

# Lecture Overview

# Pithy Quote

*If there is intelligent life on other planets, in a majority of them, they would have discovered correlated equilibrium before Nash equilibrium.*

*– Roger Myerson*

# Examples

- Consider again Battle of the Sexes.
  - Intuitively, the best outcome seems a 50-50 split between $(F, F)$ and $(B, B)$.
  - But there's no way to achieve this, so either someone loses out (unfair) or both players often miscoordinate
- Another classic example: traffic game

| | $go$ | $wait$ |
|---|---|---|
| $go$ | $-100, -100$ | $10, 0$ |
| $B$ | $0, 10$ | $-10, -10$ |

# Intuition

- What is the natural solution here?

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- What is the natural solution here?
  - A traffic light: a fair randomizing device that tells one of the agents to go and the other to wait.
- Benefits:
  - the negative payoff outcomes are completely avoided
  - fairness is achieved
  - the sum of social welfare exceeds that of any Nash equilibrium
- We could use the same idea to achieve the fair outcome in battle of the sexes.
- Our example presumed that everyone perfectly observes the random event; not required.
- More generally, some random variable with a commonly known distribution, and a private signal to each player about the outcome.
  - signal doesn't determine the outcome or others' signals; however, correlated

# Formal definition

## Definition (Correlated equilibrium)

Given an $n$-agent game $G = (N, A, u)$, a **correlated equilibrium** is a tuple $(v, \pi, \sigma)$, where $v$ is a tuple of random variables $v = (v_1, \ldots, v_n)$ with respective domains $D = (D_1, \ldots, D_n)$, $\pi$ is a joint distribution over $v$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a vector of mappings $\sigma_i : D_i \mapsto A_i$, and for each agent $i$ and every mapping $\sigma_i' : D_i \mapsto A_i$ it is the case that

$$\sum_{d \in D} \pi(d) u_i \left(\sigma_1(d_1), \ldots, \sigma_i(d_i), \ldots, \sigma_n(d_n)\right) \geq \sum_{d \in D} \pi(d) u_i \left(\sigma_1(d_1), \ldots, \sigma_i'(d_i), \ldots, \sigma_n(d_n)\right).$$

# Existence

**Theorem**

*For every Nash equilibrium $\sigma^*$ there exists a* ***corresponding correlated equilibrium*** $\sigma$.

- This is easy to show:
  - let $D_i = A_i$
  - let $\pi(d) = \prod_{i \in N} \sigma_i^*(d_i)$
  - $\sigma_i$ maps each $d_i$ to the corresponding $a_i$.
- Thus, correlated equilibria always exist

# Remarks

- Not every correlated equilibrium is equivalent to a Nash equilibrium
  - thus, correlated equilibrium is a **weaker notion** than Nash
- Any **convex combination of the payoffs** achievable under correlated equilibria is itself realizable under a correlated equilibrium
  - start with the Nash equilibria (each of which is a CE)
  - introduce a second randomizing device that selects which CE the agents will play
  - regardless of the probabilities, no agent has incentive to deviate
  - the probabilities can be adjusted to achieve any convex combination of the equilibrium payoffs
  - the randomizing devices can be combined