Game-Theoretic Analysis
Alternate Solution Concepts

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Lecture Overview

Computing Mixed Nash Equilibria

Fun Game

Maxmin and Minmax

Iterated Removal of Dominated Strategies

Rationalizability

Correlated Equilibrium
Computing Mixed Nash Equilibria: Battle of the Sexes

- It’s hard in general to compute Nash equilibria, but it’s easy when you can guess the support.
- For BoS, let’s look for an equilibrium where all actions are part of the support.
Computing Mixed Nash Equilibria: Battle of the Sexes

Let player 2 play $B$ with $p$, $F$ with $1 - p$.

If player 1 best-responds with a mixed strategy, player 2 must make her indifferent between $F$ and $B$ (why?)

\[
\begin{array}{cc}
B & F \\
B & 2, 1 & 0, 0 \\
F & 0, 0 & 1, 2 \\
\end{array}
\]
Let player 2 play \( B \) with \( p \), \( F \) with \( 1 - p \).

If player 1 best-responds with a mixed strategy, player 2 must make her indifferent between \( F \) and \( B \) (why?)

\[
\begin{align*}
    u_1(B) &= u_1(F) \\
    2p + 0(1 - p) &= 0p + 1(1 - p) \\
    p &= \frac{1}{3}
\end{align*}
\]
Computing Mixed Nash Equilibria: Battle of the Sexes

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- Likewise, player 1 must randomize to make player 2 indifferent.
  - Why is player 1 willing to randomize?
Computing Mixed Nash Equilibria: Battle of the Sexes

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- Likewise, player 1 must randomize to make player 2 indifferent.
  - Why is player 1 willing to randomize?
- Let player 1 play \( B \) with \( q \), \( F \) with \( 1 - q \).

\[
u_2(B) = u_2(F)
\]

\[
q + 0(1 - q) = 0q + 2(1 - q)
\]

\[
q = \frac{2}{3}
\]

- Thus the strategies \((\frac{2}{3}, \frac{1}{3})\), \((\frac{1}{3}, \frac{2}{3})\) are a Nash equilibrium.
Lecture Overview

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**Fun Game**

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Correlated Equilibrium
In a breakout room, play each game once as each player.

- What does row player do in equilibrium of this game?
  - row player randomizes 50–50 all the time
  - that's what it takes to make column player indifferent

- What happens when people play this game?
  - with payoff of 320, row player goes up essentially all the time
  - with payoff of 44, row player goes down essentially all the time
Fun Game!

In a breakout room, play each game once as each player.

- In an equilibrium of this game, the row player randomizes 50-50 all the time, making the column player indifferent.

- When people play this game:
  - With a payoff of 320, the row player goes up essentially all the time.
  - With a payoff of 44, the row player goes down essentially all the time.
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Unit: Lecture: Leyton-Brown & Wright (5)
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Maxmin Strategies

- Player $i$’s **maxmin strategy** is a strategy that maximizes $i$’s worst-case payoff, in the situation where all the other players (whom we denote $-i$) happen to play the strategies which cause the greatest harm to $i$.

- The **maxmin value** (or **safety level**) of the game for player $i$ is that minimum amount of payoff guaranteed by a maxmin strategy.

**Definition (Maxmin)**

The **maxmin strategy** for player $i$ is $\arg \max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$, and the **maxmin value** for player $i$ is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.

- Why would $i$ want to play a maxmin strategy?
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- Why would $i$ want to play a maxmin strategy?
  - a conservative agent maximizing worst-case payoff
  - a paranoid agent who believes everyone is out to get him
Minmax Strategies

- Player $i$’s **minmax strategy** against player $-i$ in a 2-player game is a strategy that minimizes $-i$’s best-case payoff, and the **minmax value** for $i$ against $-i$ is payoff.
- Why would $i$ want to play a minmax strategy?

**Definition (Minmax, 2-player)**

In a two-player game, the **minmax strategy** for player $i$ against player $-i$ is $\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$, and player $-i$’s **minmax value** is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$. 
Minmax Strategies

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- Why would $i$ want to play a minmax strategy?
  - to punish the other agent as much as possible

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*Unit: Lecture: Leyton-Brown & Wright (8)*
Minmax Theorem

Theorem (Minimax theorem (von Neumann, 1928))

In any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.
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In any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.

1. Each player’s maxmin value is equal to his minmax value. By convention, the maxmin value for player 1 is called the **value of the game**.
Minmax Theorem

**Theorem (Minimax theorem (von Neumann, 1928))**

*In any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.*

1. Each player’s maxmin value is equal to his minmax value. By convention, the maxmin value for player 1 is called the **value of the game**.
2. For both players, the set of maxmin strategies coincides with the set of minmax strategies.
Minmax Theorem

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In any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.

1. Each player’s maxmin value is equal to his minmax value. By convention, the maxmin value for player 1 is called the value of the game.
2. For both players, the set of maxmin strategies coincides with the set of minmax strategies.
3. Any maxmin strategy profile (or, equivalently, minmax strategy profile) is a Nash equilibrium. Furthermore, these are all the Nash equilibria. Consequently, all Nash equilibria have the same payoff vector (namely, those in which player 1 gets the value of the game).
Saddle Point: Matching Pennies

Unit: Lecture: Leyton-Brown & Wright (10)
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**Iterated Removal of Dominated Strategies**

Rationalizability

Correlated Equilibrium
Traveler’s Dilemma

Two travelers purchase identical African masks while on a tropical vacation. Their luggage is lost on the return trip, and the airline asks them to make independent claims for compensation. In anticipation of excessive claims, the airline representative announces: “We know that the bags have identical contents, and we will entertain any claim between $180 and $300, but you will each be reimbursed at an amount that equals the minimum of the two claims submitted. If the two claims differ, we will also pay a reward $R$ to the person making the smaller claim and we will deduct a penalty $R$ from the reimbursement to the person making the larger claim.”
Traveler’s Dilemma

• Action: choose an integer between 180 and 300
• If both players pick the same number, they both get that amount as payoff
• If players pick a different number:
  – the low player gets his number ($L$) plus some constant $R$
  – the high player gets $L - R$, $R = 5$.
• Play this game in a breakout room, if we have time. Do it once with $R = 5$, once with $R = 180$. 
Traveler’s Dilemma

• What is the equilibrium?

- $(180, 180)$ is the only equilibrium, for all $R \geq 2$.
- What happens?
  - with $R = 5$, most people choose 295–300
  - with $R = 180$, most people choose 180
Traveler’s Dilemma

• What is the equilibrium?
  – (180, 180) is the only equilibrium, for all \( R \geq 2 \).
Traveler’s Dilemma

- What is the equilibrium?
  - \((180, 180)\) is the only equilibrium, for all \(R \geq 2\).
- What happens?
Traveler’s Dilemma

- What is the equilibrium?
  - \((180, 180)\) is the only equilibrium, for all \(R \geq 2\).

- What happens?
  - with \(R = 5\) most people choose 295–300
  - with \(R = 180\) most people choose 180
Dominated strategies

• No equilibrium can involve a strictly dominated strategy
  – Thus we can remove it, and end up with a strategically equivalent game
  – This might allow us to remove another strategy that wasn’t dominated before
  – Running this process to termination is called iterated removal of dominated strategies.
Iterated Removal of Strictly Dominated Strategies

• This process **preserves all Nash equilibria.**
  – If there are multiple dominated strategies, the order of removal doesn’t matter
• Thus, it can be used as a **preprocessing step** before computing an equilibrium
  – Some games are solvable using this technique
  – Example: Traveler’s Dilemma!
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Correlated Equilibrium
Rationalizability

• Rather than ask what is irrational, ask what is a best response to some beliefs about the opponent
  – assumes opponent is rational
  – assumes opponent knows that you and the others are rational
  – ...

• Examples
  – is heads rationalizable in matching pennies?
Rationalizability

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• Will there always exist a rationalizable strategy?
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• Will there always exist a rationalizable strategy?
  – Yes, equilibrium strategies are always rationalizable.
Rationalizability

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- Examples
  - is heads rationalizable in matching pennies?
  - is cooperate rationalizable in prisoner’s dilemma?
- Will there always exist a rationalizable strategy?
  - Yes, equilibrium strategies are always rationalizable.
- Furthermore, in two-player games, rationalizable ⇔ survives iterated removal of strictly dominated strategies.
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Rationalizability

Correlated Equilibrium
If there is intelligent life on other planets, in a majority of them, they would have discovered correlated equilibrium before Nash equilibrium.

– Roger Myerson
Examples

- Consider again Battle of the Sexes.
  - Intuitively, the best outcome seems a 50-50 split between \((F, F)\) and \((B, B)\).
  - But there’s no way to achieve this, so either someone loses out (unfair) or both players often miscoordinate.

- Another classic example: traffic game

\[
\begin{array}{c|cc}
\text{go} & \text{wait} \\
\hline
\text{go} & -100, -100 & 10, 0 \\
B & 0, 10 & -10, -10 \\
\end{array}
\]
Intuition

• What is the natural solution here?
Intuition

- What is the natural solution here?
  - A traffic light: a fair randomizing device that tells one of the agents to go and the other to wait.

- Benefits:
  - the negative payoff outcomes are completely avoided
  - fairness is achieved
  - the sum of social welfare exceeds that of any Nash equilibrium

- We could use the same idea to achieve the fair outcome in battle of the sexes.

- Our example presumed that everyone perfectly observes the random event; not required.

- More generally, some random variable with a commonly known distribution, and a private signal to each player about the outcome.
  - signal doesn’t determine the outcome or others’ signals; however, correlated
Definition (Correlated equilibrium)

Given an \( n \)-agent game \( G = (N, A, u) \), a correlated equilibrium is a tuple \((v, \pi, \sigma)\), where \( v \) is a tuple of random variables \( v = (v_1, \ldots, v_n) \) with respective domains \( D = (D_1, \ldots, D_n) \), \( \pi \) is a joint distribution over \( v \), \( \sigma = (\sigma_1, \ldots, \sigma_n) \) is a vector of mappings \( \sigma_i : D_i \mapsto A_i \), and for each agent \( i \) and every mapping \( \sigma_i' : D_i \mapsto A_i \) it is the case that

\[
\begin{align*}
\sum_{d \in D} \pi(d) u_i (\sigma_1(d_1), \ldots, \sigma_i(d_i), \ldots, \sigma_n(d_n)) \\
\geq \sum_{d \in D} \pi(d) u_i (\sigma_1(d_1), \ldots, \sigma_i'(d_i), \ldots, \sigma_n(d_n)).
\end{align*}
\]
Existence

Theorem

For every Nash equilibrium $\sigma^*$ there exists a corresponding correlated equilibrium $\sigma$.

- This is easy to show:
  - let $D_i = A_i$
  - let $\pi(d) = \prod_{i \in N} \sigma^*_i(d_i)$
  - $\sigma_i$ maps each $d_i$ to the corresponding $a_i$.

- Thus, correlated equilibria always exist
Remarks

- Not every correlated equilibrium is equivalent to a Nash equilibrium
  - thus, correlated equilibrium is a \textit{weaker notion} than Nash
- Any \textit{convex combination of the payoffs} achievable under correlated equilibria is itself realizable under a correlated equilibrium
  - start with the Nash equilibria (each of which is a CE)
  - introduce a second randomizing device that selects which CE the agents will play
  - regardless of the probabilities, no agent has incentive to deviate
  - the probabilities can be adjusted to achieve any convex combination of the equilibrium payoffs
  - the randomizing devices can be combined