Nash Equilibrium
Game Theoretic Analysis

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Recap

Solution Concepts: Pareto Optimality

Solution Concepts: Nash Equilibrium

Mixed Strategies
Recap: Normal Form Games

In a normal form game:

- Agents **simultaneously** make a **single decision**
- They then receive an outcome that depends on the **profile of actions**

**Definition: $n$-player normal form game**

A normal form game is a tuple $G = (N, A, u)$, where

- $N$ is a set of $n$ players (indexed by $i$)
- $A = A_1 \times A_2 \times \cdots \times A_n$ is a set of action profiles
  - $A_i$ is the action set for player $i$
- $u = (u_1, \ldots, u_n)$ is a profile of utility functions
  - $u_i : A \rightarrow \mathbb{R}$
Recap: Normal Form Games as a Matrix

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- Two-player normal form games can be written as a matrix with a tuple of utilities in each cell.
- By convention, row player is first utility, column player is second utility.

Game Theoretic Analysis: Nash Equilibrium: Leyton-Brown & Wright (4)
Lecture Overview

Recap

Solution Concepts: Pareto Optimality

Solution Concepts: Nash Equilibrium

Mixed Strategies
Optimal Decisions in Games

- In single-agent decision theory, the key notion is the **optimal decision**: The decision that maximizes the agent’s expected utility:

\[
a^* = \arg \max_{a \in A} \mathbb{E}[u(a)]
\]
Optimal Decisions in Games

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• In a multiagent setting, the notion of an optimal strategy is ill-defined:

\[ a_i^* = \arg \max_{a_i \in A_i} \mathbb{E}[u_i(a_i, a_{-i})] \]
Optimal Decisions in Games

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\[ a^*_i = \arg \max_{a_i \in A_i} \mathbb{E}[u_i(a_i, a_{-i})] \]

- The best strategy depends on the strategies of the **other agents**
- But the other agents are simultaneously solving the same problem!
• From the viewpoint of an outside observer, can some outcomes of a game be considered better than others?
  – We have no justification for saying that one agent’s interests are more important than another’s
  – We cannot even compare the agents’ utilities to each other, because of affine invariance! (we don’t know what “units” the payoffs are being expressed in)

• Game theorists identify certain subsets of outcomes that are desirable and/or interesting

• These are called solution concepts
Suppose outcome $o$ is **at least as good** as $o'$ for every agent $i$

- Further, there is *some* agent who **strictly prefers** $o$ to $o'$
- E.g., $o' =$ “Everyone gets pie”, and
  $o =$ “Everyone gets pie *and also* Alice gets cake”
Pareto Optimality

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- In this situation, $o$ seems defensibly better than $o'$

**Definition**

An outcome $o^*$ is Pareto optimal if no other outcome Pareto dominates it.

**Questions**

1. Can a game have more than one Pareto-optimal outcome?
2. Does every game have at least one Pareto-optimal outcome?
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**Definition**

\( o \) **Pareto dominates** \( o' \) whenever \( o \geq_i o' \) for all \( i \in N \), and \( o >_i o' \) for some \( i \in N \)

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Game Theoretic Analysis: Nash Equilibrium: Leyton-Brown & Wright (9)
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Lecture Overview

Recap

Solution Concepts: Pareto Optimality

Solution Concepts: Nash Equilibrium

Mixed Strategies
Best Response

We can also ask: Which actions are better from an individual agent’s viewpoint?

• That depends on what the other agents are doing!
Best Response

We can also ask: Which **actions** are better from an **individual agent’s** viewpoint?

- That depends on what the other agents are doing!

**Notation**

\[
a_{-i} = (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)
\]

\[
a = (a_i, a_{-i})
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Best Response

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Definition: Best response

\[ BR_i(a_{-i}) = \{ a_i^* \in A_i \mid u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i}) \quad \forall a_i \in A_i \} \]
Nash Equilibrium

Best response is not, in itself, a solution concept

- In general, agents won’t know what the other agents will do
- But we can use it to define a solution concept called **Nash equilibrium**

A Nash equilibrium is a **stable** outcome: one where no agent regrets their action.

**Definition**

An action profile \( a \in A \) is a (pure strategy) **Nash equilibrium** iff

\[
\forall i \in N : a_i \in BR_i(a_{-i})
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Solution Concepts: Nash Equilibrium

Mixed Strategies
So far, we have been assuming that agents play a single action \textit{deterministically}.

- But we have seen that that is a pretty bad idea!
- E.g., Matching Pennies, security games

**Definition**

A strategy $s_i$ for agent $i$ is any probability distribution over the set $A_i$, where each action $a_i$ is played with probability $s_i(a_i)$. 
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So far, we have been assuming that agents play a single action *deterministically*

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- **Pure strategy:** $s_i(a_i) = 1$ for some $a_i$ *Only one action played*
- **Mixed strategy:** $s_i(a_i) < 1$ for all $a_i$ *Randomize over multiple actions*
Mixed Strategies

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- Pure strategy: $s_i(a_i) = 1$ for some $a_i$ Only one action played
- Mixed strategy: $s_i(a_i) < 1$ for all $a_i$ Randomize over multiple actions
- Set of $i$’s strategies: $S_i = \Delta(A_i)$
- Strategy profiles: $S = S_1 \times \cdots \times S_n$
Utility Under Mixed Strategies

The utility of a mixed strategy profile is its expected utility (why?)
Utility Under Mixed Strategies

The utility of a mixed strategy profile is its expected utility (why?)

1. We assume agents are decision theoretically rational
2. We assume that agents randomize independently
   \[ \Delta(A_i) \times \cdots \Delta(A_n), \text{ not } \Delta(A_i \times \cdots A_n) \]
Utility Under Mixed Strategies

The utility of a mixed strategy profile is its **expected utility** (why?)

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**Definition**

\[
    u_i(s) = \sum_{a \in A} \Pr(a \mid s)u_i(a) \\
    = \sum_{a \in A} \left( \prod_{j \in N} s_j(a_j) \right) u_i(a)
\]

*Game Theoretic Analysis: Nash Equilibrium: Leyton-Brown & Wright* (16)
Best Response and Nash Equilibrium

**Definition**

The set of $i$'s best responses to a strategy profile $s_{-i} \in S_{-i}$ is

$$BR_i(s_{-i}) = \{a_i^* \in A_i \mid u_i(a_i^*, s_{-i}) \geq u_i(a_i, s_{-i}) \quad \forall a_i \in A_i\}$$
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**Definition**

A strategy profile $s$ is a **Nash equilibrium** iff

$$\forall i \in N, s'_i \in S_i : u_i(s) \geq u_i(s'_i, s_{-i})$$

Equivalently,

$$\forall i \in N, a_i \in A_i : s_i(a_i) > 0 \iff a_i \in BR_i(s_{-i}).$$

When at least one $s_i$ is mixed, $s$ is a **mixed strategy Nash equilibrium**.
Nash’s Theorem

**Theorem [Nash 1951]**

Every game with a finite number of players and action profiles has at least one Nash equilibrium.
**Theorem** [Nash 1951]

Every game with a finite number of players and action profiles has at least one Nash equilibrium.

**Proof idea**

1. Brouwer’s fixed-point theorem guarantees that any continuous function from a simpletope to itself has at least one fixed point.
   - A simpletope is a cross product of simplices, so $S$ is a simpletope

2. Construct a continuous function $f : S \to S$ whose fixed points are all Nash equilibria
Question: Is it ever rational for an agent to play any strategy other than a Nash equilibrium strategy?

Yes!

• Even if the agent is perfectly rational, playing a Nash equilibrium strategy is only optimal if they believe that the other agents will play their parts of the same Nash equilibrium.

• Even in a zero-sum game, if you think the other agent will play in a particular sub-optimal way, a non-equilibrium strategy might be the best way to exploit them.

Example:
Lisa: Poor, predictable Bart. Always takes Rock.
Bart: Good ol’ Rock! Nothing beats Rock!
Interpreting Nash Equilibrium

**Question:** Is it ever rational for an agent to play any strategy other than a Nash equilibrium strategy? *Yes!*

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**Example**

**Lisa:** Poor, predictable Bart. Always takes Rock.

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Interpreting Mixed Strategy Nash Equilibrium

What does it even mean to say that agents are playing a mixed strategy Nash equilibrium?

- They truly are **sampling a distribution** in their heads, perhaps to confuse their opponents (e.g., zero-sum games)
- The distribution represents the **other agents’ uncertainty** about what the agent will do
- The distribution is the empirical frequency of actions in repeated play
- The distribution is the frequency of a pure strategy in a population of pure strategies
  - i.e., every individual plays a pure strategy, but individuals are sampled
Game theory studies **solution concepts** rather than simply optimal behavior

- “Optimal behavior” is not unconditionally defined in multiagent settings
- **Pareto optimal**: No agent can be made better off without making some other agent worse off
- **Nash equilibrium**: No agent regrets their strategy, given the strategies of the other agents