Utility and Foundations (2)
Modeling Human Strategic Behavior

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Lecture Overview

Recap

Proof sketch

Fun Game!
Recap: Axioms

- Completeness

\[ o_1 \succeq o_2 \text{ or } o_2 \succeq o_1 \]
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  \[ p > q \implies [p: \text{good}, (1 - p): \text{bad}] \succ [q: \text{good}, (1 - q): \text{bad}] \]

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Modeling Strategic Situations: Utility and Foundations (2): Leyton-Brown & Wright (3)
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  \[ P_{\ell_1}(o) = P_{\ell_2}(o)) \implies \ell_1 \sim \ell_2 \]

- **Continuity**
  \[ o_1 \succ o_2 \succ o_3 \implies \exists p \in [0, 1]: o_2 \sim [p: o_1, (1 - p): o_3] \]
Recap: Representation Theorem

**Theorem** [von Neumann & Morgenstern, 1944]

Suppose that a preference relation \( \succeq \) satisfies the axioms Completeness, Transitivity, Monotonicity, Substitutability, Decomposability, and Continuity.

Then there exists a function \( u : O \to \mathbb{R} \) such that

1. \( \forall o_1, o_2 \in O : o_1 \succeq o_2 \iff u(o_1) \geq u(o_2) \), and
2. \( \forall [p_1:o_1, \ldots, p_k:o_k] \in O : u([p_1:o_1, \ldots, p_k:o_k]) = \sum_{j=1}^{k} p_j u(o_j) \).

That is, there exists a utility function \( u \) that represents \( \succeq \).
Lecture Overview

Recap

Proof sketch

Fun Game!
1. Choose $o^+$, $o^-$ such that $o^- \preceq o \preceq o^+$ for all $o$. 
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   - (this turns out to be without loss of generality)
Proof sketch

1. Choose \( o^+, o^- \) such that \( o^- \preceq o \preceq o^+ \) for all \( o \)
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2. Construct \( u(o) = p \) such that \( o \sim [p:o^+, (1 - p):o^-] \)
1. Choose $o^+, o^-$ such that $o^- \preceq o \preceq o^+$ for all $o$
   - (this turns out to be without loss of generality)
2. Construct $u(o) = p$ such that $o \sim [p : o^+, (1 - p) : o^-]$
3. Substitutability lets us replace everything with these “canonical” lotteries;
   Monotonicity lets us assert the ordering between them.
Caveats & Details: Uniqueness

For a given set of preferences, the utility function is not uniquely defined.
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Comparisons of expected values are invariant to positive affine transformations:

\[ X \succeq Y \iff E[u(X)] \geq E[u(Y)] \]
Caveats & Details: Uniqueness

For a given set of preferences, the utility function is *not uniquely defined*.

Comparisons of expected values are invariant to **positive affine transformations**:

\[ X \succeq Y \iff E[u(X)] \geq E[u(Y)] \]
\[ \iff c E[u(X)] \geq c E[u(Y)] \]
\[ \iff E[cu(X)+b] \geq E[cu(Y)+b] \]

*Modeling Strategic Situations: Utility and Foundations (2): Leyton-Brown & Wright (7)*
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\[ \iff \mathbb{E}[cu(X) + b] \geq \mathbb{E}[cu(Y) + b] \]

for all \( b \in \mathbb{R} \) and \( c > 0 \)
Lecture Overview

Recap

Proof sketch

Fun Game!
Fun Game: Buying Lottery Tickets

Write down the following numbers:

1. How much would you pay to play the lottery

   \[0.3: \$5, 0.3: \$7, 0.4: \$9\]?
Fun Game: Buying Lottery Tickets

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\[0.3 : $5, \ 0.3 : $7, \ 0.4 : $9\]?

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\[p : $5, \ q : $7, \ (1 - p - q) : $9\]?
Fun Game: Buying Lottery Tickets

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2. How much would you pay to play the lottery

   \([p: \$5, \ q: \$7, \ (1 - p - q): \$9]\)?

3. How much would you pay to play the lottery

   \([p: \$5, \ q: \$7, \ (1 - p - q): \$9]\)

   *If you knew that the last seven draws had been 5, 5, 7, 5, 9, 9, 5?*
Beyond von Neumann & Morgenstern

• The first game was a pretty good match for the utility theory that we just learned.
• **Question:** If two rational agents have different prices for [0.3 : $5, 0.3 : $7, 0.4 : $9], what does that suggest about their preferences for money?
Beyond von Neumann & Morgenstern

- The first game was a pretty good match for the utility theory that we just learned.
- **Question:** If two rational agents have different prices for $[0.3 : $5, 0.3 : $7, 0.4 : $9]$, what does that suggest about their preferences for money?
- The second game was not such a great match!
- **Question:** If two rational agents have different prices for $[p : $5, q : $7, (1 − p − q) : $9]$, can we infer anything about the two agents’ preferences for money?
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- **Question:** If two rational agents have different prices for \([p:5, q:7, (1 - p - q):9]\), can we infer anything about the two agents’ preferences for money?
- If the two agents agree about the price for \([p:5, q:7, (1 - p - q):9]\) but then disagree once they hear what the last few draws were?
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- The first game was a pretty good match for the utility theory that we just learned.
- **Question:** If two rational agents have different prices for \([0.3 : $5, 0.3 : $7, 0.4 : $9]\), what does that suggest about their **preferences for money**?
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- **Question:** If two rational agents have different prices for \([p : $5, q : $7, (1 - p - q) : $9]\), can we infer anything about the two agents’ **preferences for money**?
- If the two agents agree about the price for \([p : $5, q : $7, (1 - p - q) : $9]\) but then disagree once they hear what the last few draws were?
- von Neumann and Morgenstern’s utility theory assumes **known, objective** probabilities.
- There are other representation theorems [e.g., Savage 1954] that state that rational agents must (a) have probabilistic beliefs, (b) update those beliefs as if by conditioning, (c) maximize the expected value of some utility function wrt them.

*Modeling Strategic Situations: Utility and Foundations (2): Leyton-Brown & Wright (10)*
Utility theory proves that agents whose preferences obey certain simple axioms about preferences over lotteries must act as if they were maximizing the expected value of a scalar function.

• “Rational” agents are those whose behaviour satisfies the axioms

• If you don’t buy the axioms, then you shouldn’t buy that this theorem is about rational behavior.

• Conversely, if you don’t buy that rational agents must behave in this way, then there must be at least one axiom that you disagree with.

This approach extends to “subjective” probabilities:

• Axioms about preferences over uncertain “acts” that do not describe how agents manipulate probabilities.
Game Representations

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Normal-Form

Repeated

Extensive Form

Bayesian Games
Should you send your packets using correctly-implemented TCP (which has a “backoff” mechanism) or using a defective implementation (which doesn’t)?

- Consider this situation as a two-player game:
  - both use a correct implementation: both get 1 ms delay
  - one correct, one defective: 4 ms delay for correct, 0 ms for defective
  - both defective: both get a 3 ms delay.
TCP Backoff Game

- Consider this situation as a two-player game:
  - both use a correct implementation: both get 1 ms delay
  - one correct, one defective: 4 ms delay for correct, 0 ms for defective
  - both defective: both get a 3 ms delay.

- Go into a breakout room. Play once with each person.

- Questions:
  - What action should a player of the game take?
  - Would all users behave the same in this scenario?
  - What global patterns of behaviour should the system designer expect?
  - Under what changes to the delay numbers would behavior be the same?
  - What effect would communication have?
  - Does it matter if I believe that my opponent is rational?
Defining Games

• Finite, $n$-person game: $\langle N, A, u \rangle$:
  – $N$ is a finite set of $n$ players, indexed by $i$
  – $A = \langle A_1, \ldots, A_n \rangle$ is a tuple of action sets for each player $i$
    • $a \in A$ is an action profile
  – $u = \langle u_1, \ldots, u_n \rangle$, a utility function for each player, where $u_i : A \rightarrow \mathbb{R}$

• Writing a 2-player game as a matrix:
  – row player is player 1, column player is player 2
  – rows are actions $a \in A_1$, columns are $a' \in A_2$
  – cells are outcomes, written as a tuple of utility values for each player
Games in Matrix Form

Here's the **TCP Backoff Game** written as a matrix ("normal form").

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>-1, -1</td>
<td>-4, 0</td>
</tr>
<tr>
<td>D</td>
<td>0, -4</td>
<td>-3, -3</td>
</tr>
</tbody>
</table>
Prisoner's dilemma is any game

\[
\begin{array}{cc}
C & D \\
C & a, a & b, c \\
D & c, b & d, d \\
\end{array}
\]

with \( c > a > d > b \).
Players have **exactly opposed** interests

- There must be precisely two players *(otherwise they can’t have exactly opposed interests)*
- For all action profiles \( a \in A, u_1(a) + u_2(a) = c \) for some constant \( c \)
  - Special case: zero sum
- Thus, we only need to store a utility function for one player
  - In a sense, it’s a one-player game
One player wants to **match**; the other wants to **mismatch**.

<table>
<thead>
<tr>
<th></th>
<th>Heads</th>
<th>Tails</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heads</td>
<td>1, -1</td>
<td>-1, 1</td>
</tr>
<tr>
<td>Tails</td>
<td>-1, 1</td>
<td>1, -1</td>
</tr>
</tbody>
</table>
Generalized matching pennies.

<table>
<thead>
<tr>
<th></th>
<th>Rock</th>
<th>Paper</th>
<th>Scissors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rock</td>
<td>0,0</td>
<td>-1,1</td>
<td>1, -1</td>
</tr>
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</table>

...Believe it or not, there’s an annual international competition for this game!
Players have \textit{exactly the same} interests.

- no conflict: all players want the same things
- \( \forall a \in A, \forall i, j, u_i(a) = u_j(a) \)
- we often write such games with a single payoff per cell
- why are these even still games?
Which **side of the road** should you drive on?

<table>
<thead>
<tr>
<th></th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left</td>
<td>1,1</td>
<td>0,0</td>
</tr>
<tr>
<td>Right</td>
<td>0,0</td>
<td>1,1</td>
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</tbody>
</table>
The most interesting games combine elements of cooperation and competition.

![Battle of the Sexes Game](image-url)
The most interesting games combine elements of cooperation and competition.

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>F</td>
<td>0, 0</td>
<td>1, 2</td>
</tr>
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Play this game in breakout rooms. Be fast!