Bayesian Games
Game Theoretic Analysis

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Lecture Overview

Representing Bayesian Games

Analyzing Bayesian games

Recall our Previous Fun Game

- Choose a phone number none of your neighbours knows; consider it to be ABC-DEFG
  - take "DE" as your valuation
  - play a first-price auction with three neighbours, where your utility is your valuation minus the amount you pay
  - now play the auction again, same neighbours, same valuation
  - now play again, with "FG" as your valuation

- Let's reflect again on what happened when we played the game
  - what is the role of uncertainty here?
  - can we model this uncertainty using an imperfect information extensive form game?

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  - can we model this uncertainty using an imperfect information extensive form game?
    - imperfect info means not knowing what node you’re in in the info set

Definition 1: Information Sets

- **Bayesian game**: a set of games that differ only in their payoffs, a common prior defined over them, and a partition structure over the games for each agent.

Definition (Bayesian Game: Information Sets)

A **Bayesian game** is a tuple (N, G, P, I) where

- N is a set of agents,
- G is a set of games with N agents each such that if $g, g' \in G$ then for each agent $i \in N$ the strategy space in g is identical to the strategy space in g',
- $P \in \Pi(G)$ is a common prior over games, where $\Pi(G)$ is the set of all probability distributions over G, and
- $I = (I_1, ..., I_N)$ is a set of partitions of G, one for each agent.

Definition 1: Example

$I_{2,1}$ | $I_{2,2}$

$I_{1,1}$

MP

$2, 0$	$0, 2$
$0, 2$	$2, 0$

$p = 0.3$

PD

$2, 2$	$0, 3$
$3, 0$	$1, 1$

$p = 0.1$

$I_{1,2}$

Coord

$2, 2$	$0, 0$
$0, 0$	$1, 1$

$p = 0.2$

BoS

$2, 1$	$0, 0$
$0, 0$	$1, 2$

$p = 0.4$

Definition 2: Epistemic Types

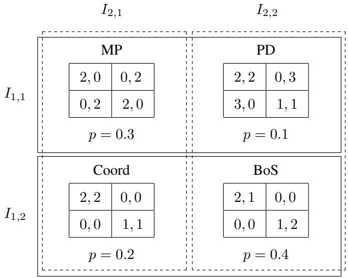
- Directly represent uncertainty over utility function using the notion of **epistemic type**.

Definition

A **Bayesian game** is a tuple (N, A, Θ, p, u) where

- N is a set of agents,
- $A = (A_1, \ldots, A_n)$, where A_i is the set of actions available to player i,
- $\Theta = (\Theta_1, \ldots, \Theta_n)$, where Θ_i is the type space of player i,
- $p : \Theta \to [0, 1]$ is the common prior over types,
- $u = (u_1, \ldots, u_n)$, where $u_i : A \times \Theta \to \mathbb{R}$ is the utility function for player i.

Definition 2: Example



a_1	a_2	θ_1	θ_2	u_1	u_2
U	L	$\theta_{1,1}$	$\theta_{2,1}$	2	0
U	L	$\theta_{1,1}$	$\theta_{2,2}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,1}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	2	1
U	R	$\theta_{1,1}$	$\theta_{2,1}$	0	2
U	R	$\theta_{1,1}$	$\theta_{2,2}$	0	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	0	0
U	R	$\theta_{1,2}$	$\theta_{2,2}$	0	0

a_1	a_2	θ_1	θ_2	u_1	u_2
D	L	$\theta_{1,1}$	$\theta_{2,1}$	0	2
D	L	$\theta_{1,1}$	$\theta_{2,2}$	3	0
D	L	$\theta_{1,2}$	$\theta_{2,1}$	0	0
D	L	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	R	$\theta_{1,1}$	$\theta_{2,1}$	2	0
D	R	$\theta_{1,1}$	$\theta_{2,2}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,1}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,2}$	1	2

Fun Game 2: Chicken... after dark!

	G	S
G	$\theta_1 - \theta_2 - 2, \theta_2 - \theta_1 - 2$	$3, -2$
S	$-2, 3$	$-1, -1$

- Write down the numbers 0, 1, 2, 10 on 4 pieces of paper. This is the deck of cards.
- Each player draws 1 card (the size/power of your car).
- Play chicken! If you collide, each player's utility depends on the size of both cars.

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- This game is a bit like poker. What's missing?

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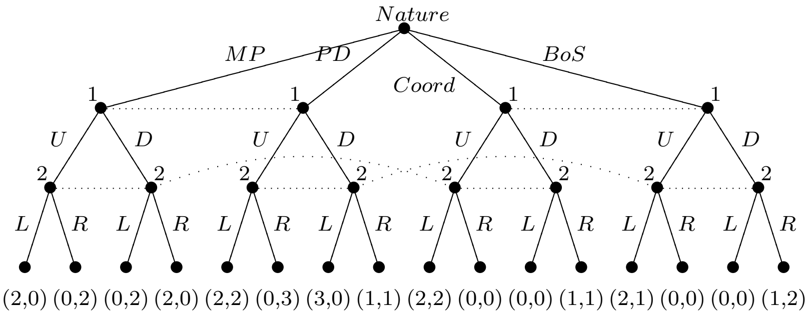
- Each player draws 1 card (the size/power of your car).
- Play chicken! If you collide, each player's utility depends on the size of both cars.
- This game is a bit like poker. What's missing? Learning anything about your opponent's private information.

Definition 3: Extensive Form with Chance Moves

- Add an agent, “Nature,” who follows a commonly known mixed strategy.
- Thus, reduce Bayesian games to extensive form games of imperfect information.
- This definition is cumbersome for the same reason that IIEF is a cumbersome way of representing matrix games like Prisoner’s dilemma
  - however, it makes sense when the agents really do move sequentially, and at least occasionally observe each other’s actions.

Definition 3: Example



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Analyzing Bayesian games

Strategies

- **Pure strategy**: $s_i : \Theta_i \rightarrow A_i$
  - a mapping from every type agent i could have to the action he would play if he had that type.
- **Mixed strategy**: $s_i : \Theta_i \rightarrow \Pi(A_i)$
  - a mapping from i's type to a probability distribution over his action choices.
- $s_j(a_j|\theta_j)$
  - the probability under mixed strategy s_j that agent j plays action a_j, given that j's type is θ_j.
- Notions like dominance still apply.

Expected Utility

Three meaningful notions of expected utility:

- ***ex-ante***
  - the agent knows nothing about anyone's actual type;
- ***ex-interim***
  - an agent knows his own type but not the types of the other agents;
- ***ex-post***
  - the agent knows all agents' types.

Ex-interim expected utility

Definition (*Ex-interim* expected utility)

Agent i's ***ex-interim* expected utility** in a Bayesian game (N, A, Θ, p, u), where i's type is θ_i and where the agents' strategies are given by the mixed strategy profile s, is defined as

$$EU_i(s|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a, \theta_{-i}, \theta_i).$$

- i must consider every θ_{-i} and every a in order to evaluate $u_i(a, \theta_i, \theta_{-i})$.
- i must weight this utility value by:
  - the probability that a would be realized given all players' mixed strategies and types;
  - the probability that the other players' types would be θ_{-i} given that his own type is θ_i.

Ex-ante expected utility

Definition (*Ex-ante* expected utility)

Agent i's ***ex-ante* expected utility** in a Bayesian game (N, A, Θ, p, u), where the agents' strategies are given by the mixed strategy profile s, is defined as

$$EU_i(s) = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s|\theta_i)$$

or equivalently as

$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a, \theta).$$

Ex-post expected utility

Definition (*Ex-post* expected utility)

Agent i's ***ex-post* expected utility** in a Bayesian game (N, A, Θ, p, u), where the agents' strategies are given by s and the agent' types are given by θ, is defined as

$$EU_i(s, \theta) = \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a, \theta).$$

- The only uncertainty here concerns the other agents' mixed strategies, since i knows everyone's type.

Best response

Definition (Best response in a Bayesian game)

The set of agent i's **best responses** to mixed strategy profile s_{-i} are given by

$$BR_i(s_{-i}) = \arg\max_{s'_i \in S_i} EU_i(s'_i, s_{-i}).$$

- it may seem odd that BR is calculated based on i's *ex-ante* expected utility.
- However, write $EU_i(s)$ as $\sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s|\theta_i)$ and observe that $EU_i(s'_i, s_{-i}|\theta_i)$ does not depend on strategies that i would play if his type were not θ_i.
- Thus, we are in fact performing independent maximization of i's *ex-interim* expected utility conditioned on each type that he could have.

Nash equilibrium

Definition (Bayes-Nash equilibrium)

A **Bayes-Nash equilibrium** is a mixed strategy profile s that satisfies $\forall i \;\; s_i \in BR_i(s_{-i})$.

- we can also construct an induced normal form for Bayesian games
- the numbers in the cells will correspond to ***ex-ante*** expected utilities
  - however as argued above, as long as the strategy space is unchanged, best responses don't change between the *ex-ante* and *ex-interim* cases.

ex-post Equilibrium

Definition (*ex-post* equilibrium)

A ***ex-post* equilibrium** is a mixed strategy profile s that satisfies $\forall\theta, \ \forall i$,
$s_i \in \arg\max_{s_i' \in S_i} EU_i(s_i', s_{-i}, \theta)$.

- somewhat similar to **dominant strategy**, but not quite
  - EP: agents do not need to have accurate beliefs about the type distribution
  - DS: agents do not need to have accurate beliefs about others' strategies