Imperfect Information Extensive Form Games

Game Theoretic Analysis

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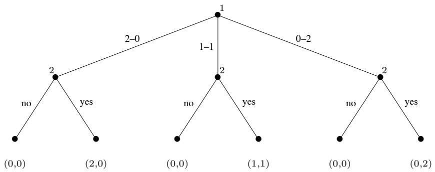
Perfect Information Extensive Form Game

Definition

A **finite perfect information game in extensive form** is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u)$, where

- N is a set of n players
- A is a single set of actions
- H is a set of nonterminal choice nodes
- Z is a set of terminal nodes (disjoint from H)
- $\chi : H \to 2^A$ is the action function
- $\rho : H \to N$ is the player function
- $\sigma : H \times A \to H \cup Z$ is the successor function
- $u = (u_1, \ldots, u_n)$ is a profile of utility functions $u_i : Z \to \mathbb{R}$ for each player i



Recap: Pure Strategies

Definition

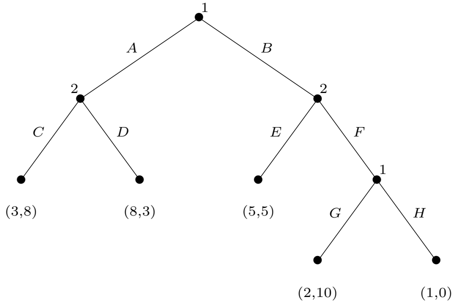
Let $G = (N, A, H, Z, \chi, \rho, \sigma, u)$ be a perfect information game in extensive form. Then the **pure strategies** for player i consist of the cross product of actions available to i at each of their choice nodes:

$$\prod_{h \in H | \rho(h) = o} \chi(h).$$

Note that a pure strategy associates an action with **every** choice node, even those that will **never be reached**.

Recap: Induced Normal Form

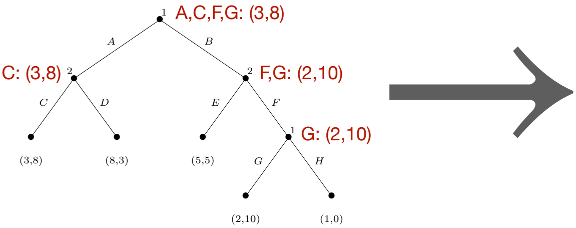
- Any pair of pure strategies uniquely identifies a **terminal node**, which identifies a **utility** for each agent
- We have now defined a set of agents, pure strategies, and utility functions
- Any perfect-information extensive form game defines a corresponding **induced normal form game**



	C, E	C, F	D, E	D, F
A, G	3,8	3,8	8,3	8,3
A, H	3,8	3,8	8,3	8,3
B, G	5,5	2,10	5,5	2,10
B, H	5,5	1,0	5,5	1,0

Recap: Backward Induction

- Backward induction is a straightforward algorithm that is guaranteed to compute a **pure strategy, subgame perfect equilbrium**
- **Idea:** Replace subgames with their equilibrium values



$$(A, G), (C, F)$$

Lecture Overview

Imperfect Information Games

Behavioural vs. Mixed Strategies

Perfect vs. Imperfect Recall

Imperfect Information, informally

- Perfect information extensive form games model sequential actions that are **observed by all players**

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- Perfect information extensive form games model sequential actions that are **observed by all players**
- But many games involve **hidden actions**
  - Cribbage, poker, Scrabble
  - Sometimes the actions of the players are hidden, sometimes "Nature's" actions are hidden, sometimes both

Imperfect Information, informally

- Perfect information extensive form games model sequential actions that are **observed by all players**
- But many games involve **hidden actions**
  - Cribbage, poker, Scrabble
  - Sometimes the actions of the players are hidden, sometimes "Nature's" actions are hidden, sometimes both
- Imperfect information extensive form games are a model of games with sequential actions, some of which may be **hidden**

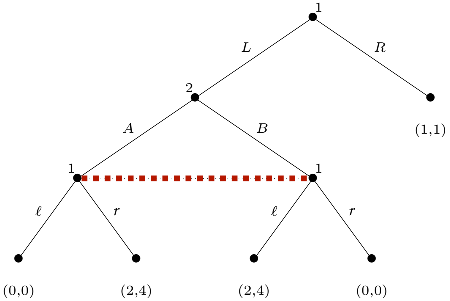
Imperfect Information Extensive Form Game

Definition

An imperfect information game in extensive form is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u, \mathcal{I})$, where

- $(N, A, H, Z, \chi, \rho, \sigma, u)$ is a perfect information extensive form game, and
- $\mathcal{I} = (I_1, \ldots, I_n)$, where for each $i \in N$, $\mathcal{I}_i = (I_{i,1}, \ldots, I_{i,k_i})$ is an **equivalence relation** on (i.e., partition of) $\{h \in H \mid \rho(h) = i\}$, and
- For every h, h' such that $h \in I$ and $h' \in I$ for some $i \in N$ and $I \in \mathcal{I}_i$, $\rho(h) = \rho(h') = i$ and $\chi(h) = \chi(h')$.

Imperfect Information EFG Example



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- Players cannot distinguish which history they are in within an information set

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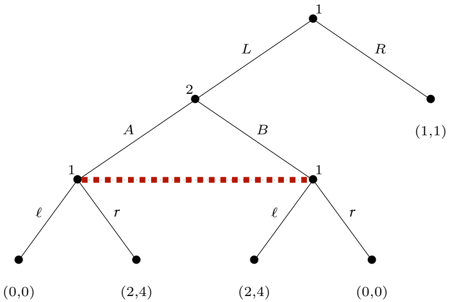

- The elements of the partition are often called **information sets**
- Players cannot distinguish which history they are in within an information set
- **Question:** What are the information sets for each player in this game?

Pure Strategies

Question: What are **pure strategies** in an imperfect information extensive form game?

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Definition

Let $G = (N, A, H, Z, \chi, \rho, \sigma, u, \mathcal{I})$ be an imperfect information game in extensive form. Then the set of **pure strategies of player i** is the cross product of actions available to player i at each of their **information sets**, i.e.,

$$\prod_{I \in \mathcal{I}_i} \chi(I),$$

where $\chi(I) = \chi(h)$ for an arbitrary $h \in I$.

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In an imperfect information EFG:

1. What are the mixed strategies?

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2. What is a best response?

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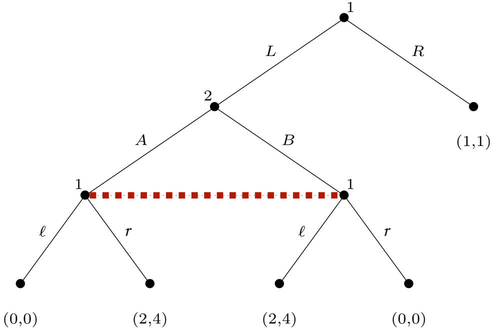
Questions

In an imperfect information EFG:

1. What are the mixed strategies?
2. What is a best response?
3. What is a Nash equilibrium?

Induced Normal Form



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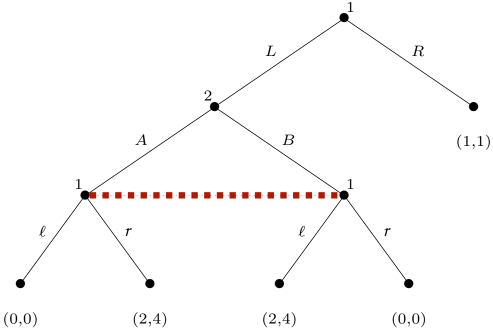

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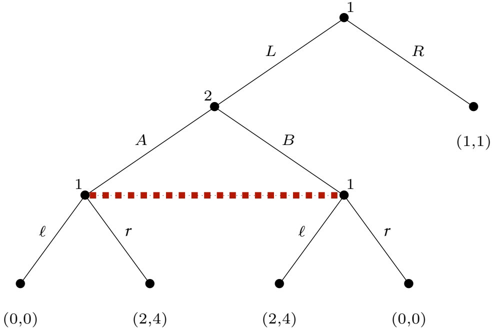

	A	B
L,ℓ	0,0	2,4
L,r	2,4	0,0
R,ℓ	1,1	1,1
R,r	1,1	1,1

G

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- Any imperfect information extensive form game defines a corresponding **induced normal form game**

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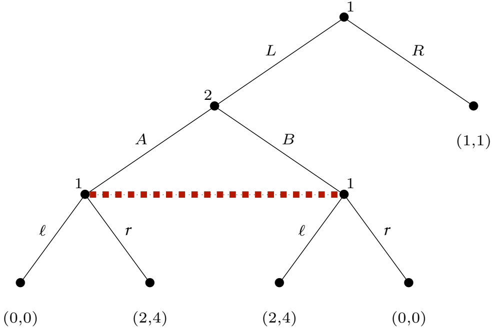

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G

Question

Can you represent an arbitrary perfect information EFG as an imperfect information EFG?

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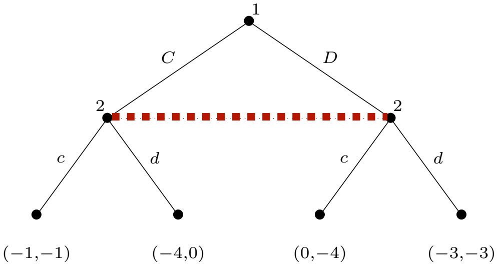
Normal to Extensive Form

	c	d
C	-1,-1	-4,0
D	0,-4	-3,-3

G



- Unlike perfect information EFGs, we can also represent **any normal form game** as an imperfect infomation EFG
- Players can play in **any order** (**why?**)

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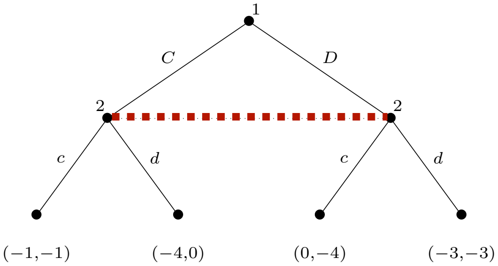

- Unlike perfect information EFGs, we can also represent **any normal form game** as an imperfect infomation EFG
- Players can play in **any order** (**why?**)
- **Question:** What happens if we run this NFG→EFG translation on the induced normal form of an arbitrary extensive form game?

Lecture Overview

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Behavioural vs. Mixed Strategies

Perfect vs. Imperfect Recall

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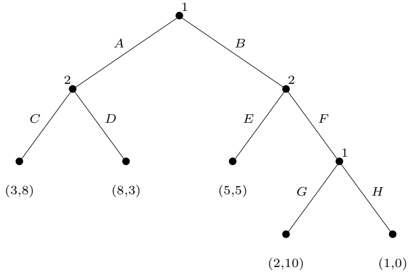
Definition

A **behavioral strategy** b_i is a mapping from a an agent's information sets to a distribution over the actions at that information set, which is *sampled independently each time* the agent arrives at the information set:

$$b_i \in [\Delta(\chi(I))]_{I \in \mathcal{I}_i}$$

Behavioural vs. Mixed Example

- Behavioural strategy: $([.6 : A,\ .4 : B], [.6 : G,\ .4 : H])$
- Mixed strategy: $[.6 : (A, G),\ .4 : (B, H)]$



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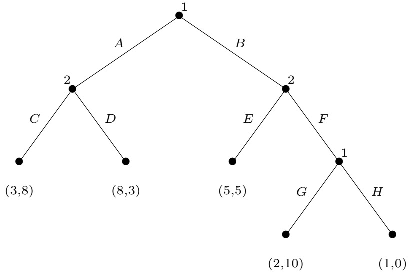

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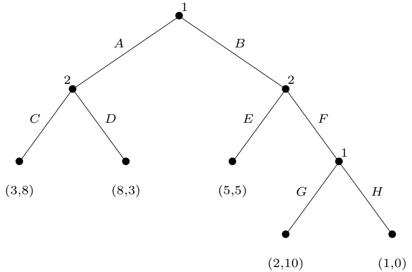

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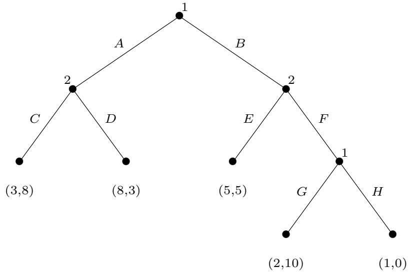

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Perfect vs. Imperfect Recall

Perfect Recall

Definition

Player i has **perfect recall** in an imperfect information game G if for any two choice nodes $h, h' \in I$ that are in the same information set $I \in \mathcal{I}_i$ for player i, for any path $h_0, a_0, h_1, a_1, \ldots, h_n, h$ from the root of the game to h, and for any path $h'_0, a'_0, h'_1, a'_1, \ldots, h'_m, h'$ from the root of the game to h', it must be the case that:

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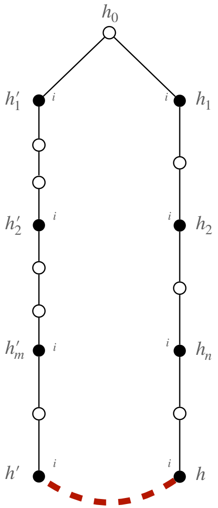

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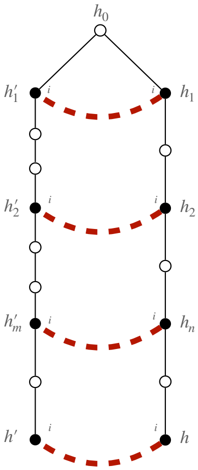

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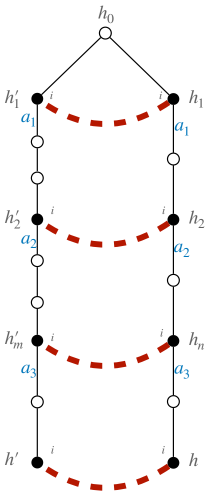

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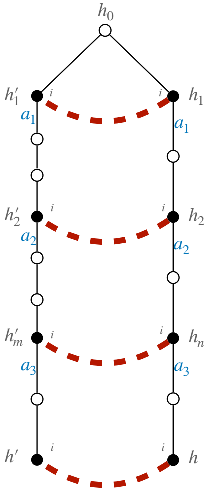
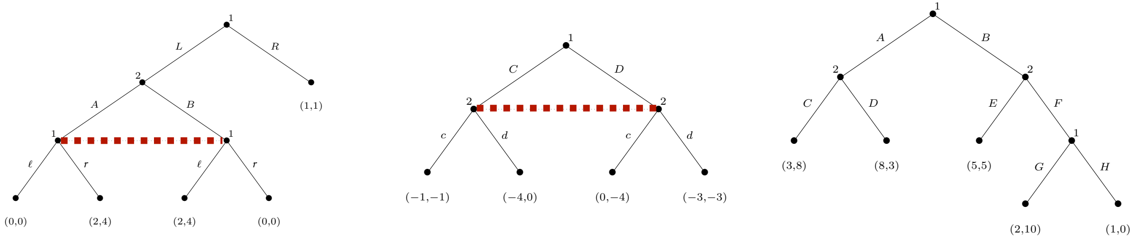
3. for all $0 \leq j \leq n$, if $\rho(h_j) = i$, then $a_j = a'_j$.

G is a **game of perfect recall** if every player has perfect recall in G.



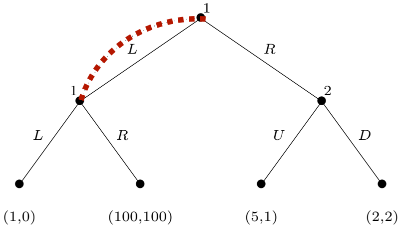
Perfect Recall Examples



Question

Which of the above games is a game of **perfect recall**?

Imperfect Recall Example



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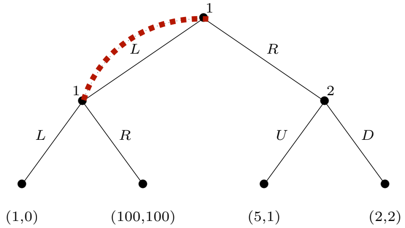

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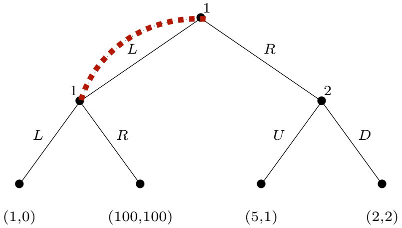

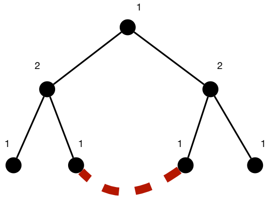

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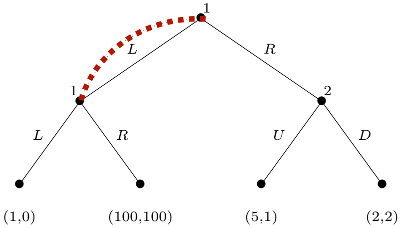

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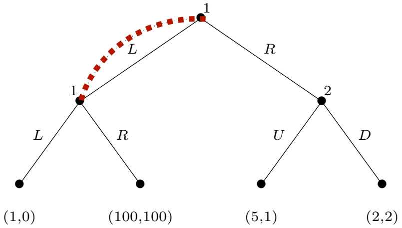

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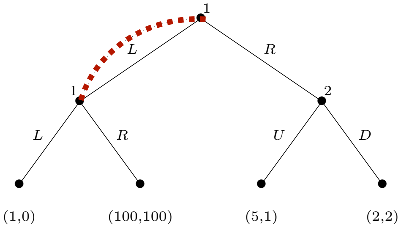
- **Question:** What is the **mixed strategy equilibrium** of this game?



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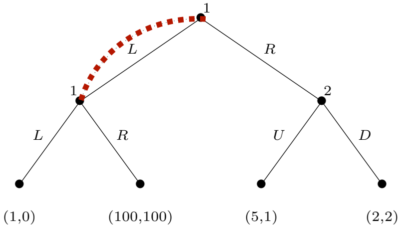
- **Question:** What is the **equilibrium in behavioural strategies** of this game?



Imperfect Recall Applications

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1. When the **actual agents** may forget previous history
   - Including cases where the agent strategies are really executed by proxies
2. As an **approximation technique**
   - E.g., poker: The exact exact cards that have been played to this point may not matter as much as some coarse grouping of which cards have been played
   - Grouping the cards into equivalence classes is a **lossy** approximation

Kuhn's Theorem

Theorem [Kuhn, 1953]

In a game of perfect recall, any mixed strategy of a given agent can be replaced by an **equivalent behavioural strategy**, and any **behavioural strategy** can be replaced by an equivalent mixed strategy.

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Corollary

Restricting attention to behavioral strategies does not change the set of Nash equilibria in a game of perfect recall. (**why?**)

Summary

- **Imperfect information extensive form games** are a model of games with sequential actions, some of which may be **hidden**
  - Histories are partitioned into information sets
  - Players **cannot distinguish** between histories in the same information set
- **Pure strategies** map each information set to an action
  - Mixed strategies are distributions over **pure strategies**
  - Behavioural strategies map each information set to a distribution over **actions**
  - In games of perfect recall, mixed strategies and behavioral strategies are interchangeable