Extensive Form Games

Game Theoretic Analysis

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Recap: Best Response & Nash Equilibrium

Definition

The set of i's best responses to a strategy profile $s_{-i} \in S_{-i}$ is

$$BR_i(s_{-i}) = \{a_i^* \in A_i \mid u_i(a_i^*, s_{-i}) \geq u_i(a_i, s_{-i}) \quad \forall a_i \in A_i\}$$

Definition

A strategy profile s is a **Nash equilibrium** iff

$$\forall i \in N, s_i' \in S_i : u_i(s) \geq u_i(s_i', s_{-i})$$

Equivalently,

$$\forall i \in N, a_i \in A_i : s_i(a_i) > 0 \iff a_i \in BR_i(s_{-i}).$$

When at least one s_i is mixed, s is a mixed strategy Nash equilibrium

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Recap: Rationalizability

A rationalizable strategy is one which is a best response to some belief about the other agents

- that also assumes opponent is playing some rationalizable strategy
- the beliefs need not be consistent with each other

In two-player games, rationalizable strategies are exactly those that survive **iterated removal of strictly dominated strategies**.

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Example: Traveller's Dilemma

- 300 is **weakly dominated** by 299
- But it is **strictly dominated** by a mixed strategy over the actions 180–299.
- So 300 does not survivie iterated removal of strictly dominated strategies
- In the game with 300 removed, 299 is weakly dominated by 298
- ...but **strictly dominated** by a mixed strategy over 180–298

Lecture Overview

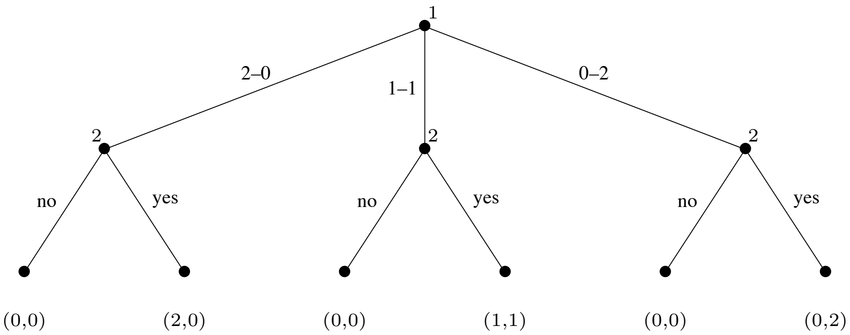
Extensive Form Games

Nash equilibrium

Subgame Perfect Equilibrium

Extensive Form Games

- Normal form games don't have any notion of *sequence*: all actions happen **simultaneously**
- The extensive form is a game representation that explicitly includes temporal structure (i.e., a game tree)



Perfect Information

There are two kinds of extensive form game:

1. **Perfect information:** Every agent **sees all actions** of the other players (including any special "Chance" player)
   - e.g., Chess, Checkers, Backgammon, Pandemic
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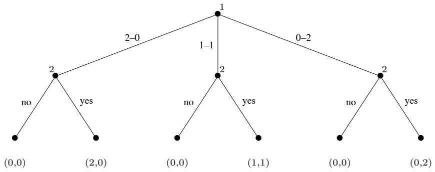
   - e.g., Chess, Checkers, Backgammon, Pandemic
   - This lecture!
2. **Imperfect information:** Some actions are **hidden**
   - Players may not know exactly where they are in the tree
   - Different players may have different knowledge (about where they are in the tree)
   - E.g., Poker, Rummy, Scrabble

Perfect Information Extensive Form Game

Definition

A **finite perfect information game in extensive form** is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u)$, where

- N is a set of n **players**
- A is a single set of **actions**



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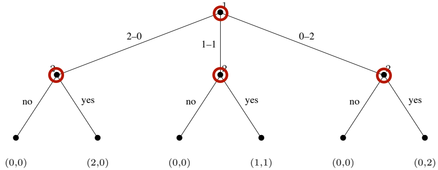

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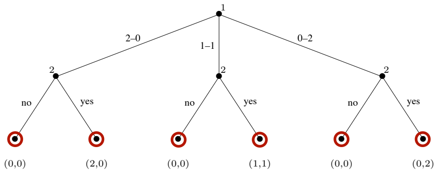

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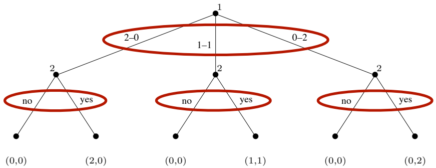

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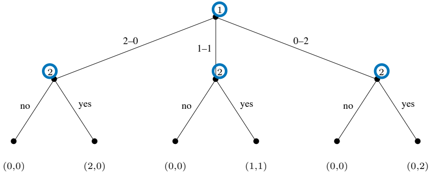

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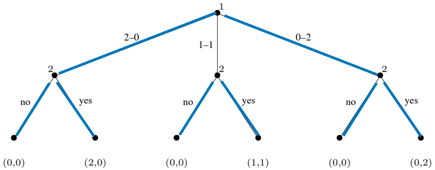
- $\sigma : H \times A \to H \cup Z$ is the **successor function**



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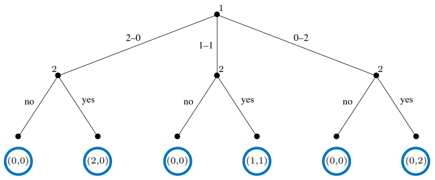
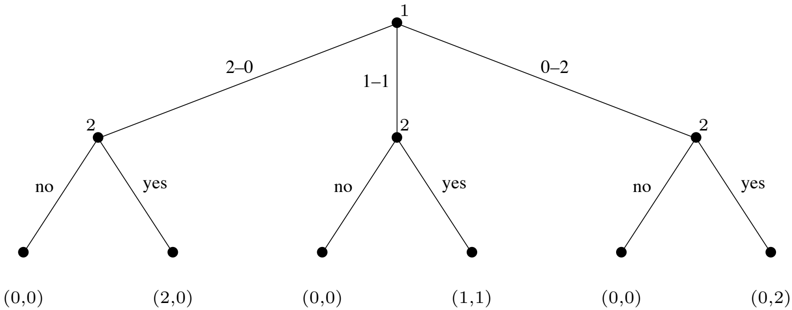
- $\sigma : H \times A \to H \cup Z$ is the successor function
- $u = (u_1, \ldots, u_n)$ is a profile of **utility functions** $u_i : Z \to \mathbb{R}$ for each player i



Fun Game: The Sharing Game



- Two siblings must decide how to share two $100 coins
- Sibling 1 suggests a division, then sibling 2 accepts or rejects
  - If rejected, nobody gets any coins
- Play against 2 other people, once per person, different role each time

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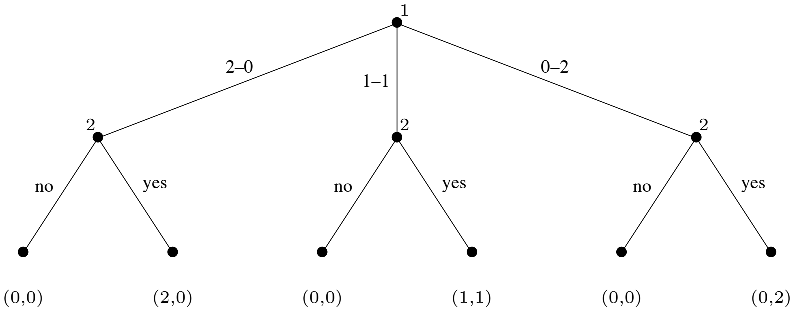

- Two siblings must decide how to share two $100 coins
- Sibling 1 suggests a division, then sibling 2 accepts or rejects
  - If rejected, nobody gets any coins
- Play against 2 other people, once per person, different role each time
- **Question:** Did you have a plan for every possible eventuality?

Lecture Overview

Extensive Form Games

Nash equilibrium

Subgame Perfect Equilibrium

Pure Strategies

Question

What are the **pure strategies** in an extensive form game?

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What are the **pure strategies** in an extensive form game?

Definition

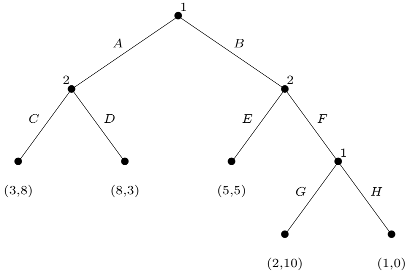
Let $G = (N, A, H, Z, \chi, \rho, \sigma, u)$ be a perfect information game in extensive form. Then the **pure strategies** for player i consist of the cross product of actions available to i at each of their choice nodes:

$$\prod_{h \in H | \rho(h) = o} \chi(h).$$

Note that a pure strategy associates an action with **every** choice node, even those that will **never be reached**.

Pure Strategies Example

Question: What are the pure strategies for **player 2**?



Pure Strategies Example

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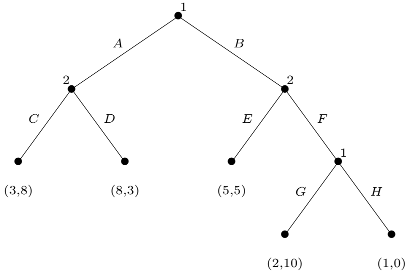
$$\{(C, E), (C, F), (D, E), (D, F)\}$$



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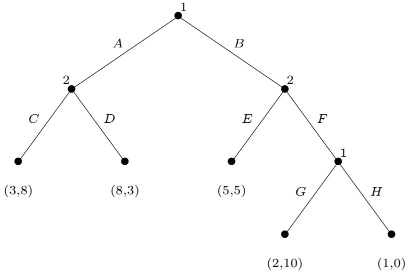

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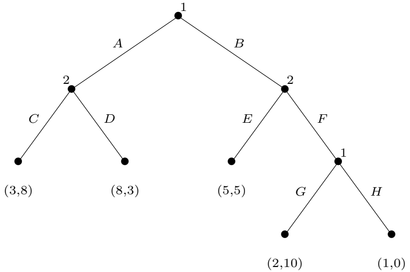
Question: What are the pure strategies for **player 1**?

$$\{(A,G),(A,H),(B,G),(B,H)\}$$



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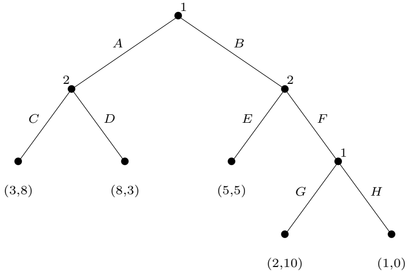
Question: What are the pure strategies for **player 1**?

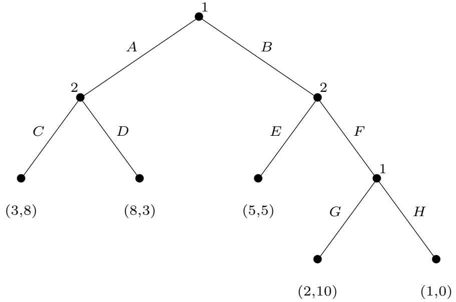
$$\{(A, G), (A, H), (B, G), (B, H)\}$$

Note that there is always an action for the second node, even when it cannot be reached.



Induced Normal Form

- Any pair of pure strategies uniquely identifies a **terminal node**, which identifies a **utility** for each agent (**why?**)



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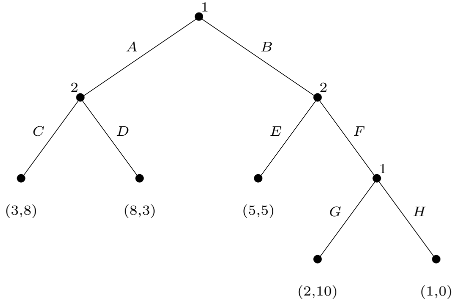
- Any perfect-information extensive form game defines a corresponding **induced normal form game**



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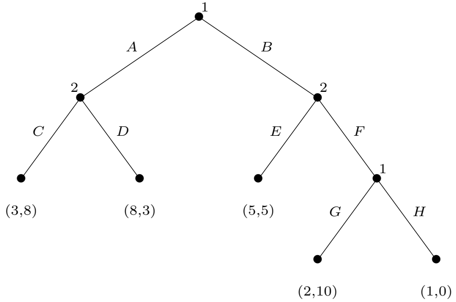
- **Question:** Which representation is more **compact**?



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Reusing Old Definitions

We can also plug our new definition of pure strategy into our existing definitions for:

- Mixed strategy
- Best response
- Nash equilibrium (both pure strategy and mixed strategy)

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- Mixed strategy
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Question

What is the definition of a **mixed strategy** in an extensive form game?

Pure Strategy Nash Equilibria

Theorem [Zermelo, 1913]

Every finite perfect-information game in extensive form has at least one **pure strategy Nash equilibrium**.

Pure Strategy Nash Equilibria

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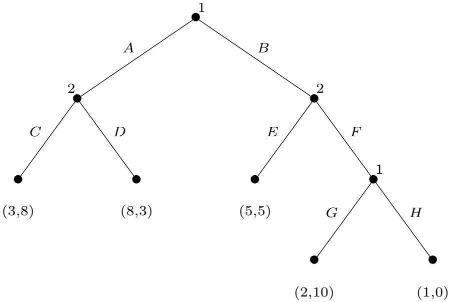
Every finite perfect-information game in extensive form has at least one **pure strategy Nash equilibrium**.

Proof: Solve by **backward induction**

- Starting from the bottom of the tree, no agent needs to randomize, because there is a deterministic best response.
- Replace those nodes with the resulting utility vector
- Repeat until an action is assigned for all choice nodes

(There might be multiple pure strategy Nash equilibria in cases where an agent has multiple best responses at a single choice node.)

Pure Strategy Nash Equilibria Example



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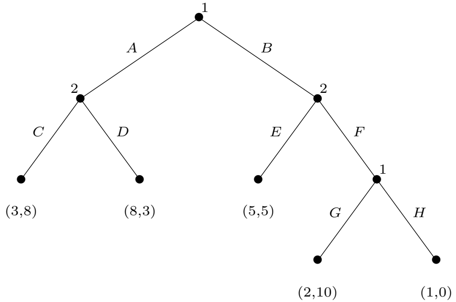

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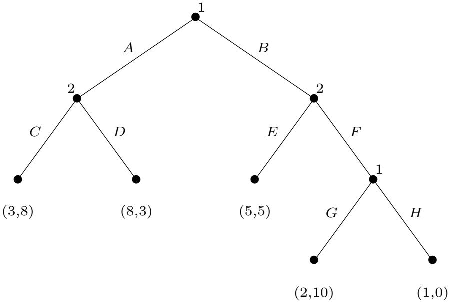

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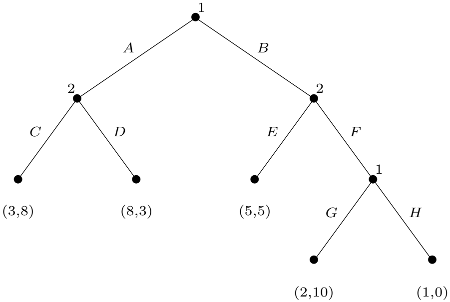

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Lecture Overview

Extensive Form Games

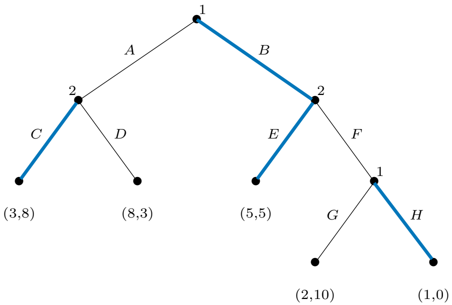
Nash equilibrium

Subgame Perfect Equilibrium

Subgame Perfection, informally

Some equilibria seem **less plausible** than others.

- $\langle (B,H),(C,E)\rangle$: F has payoff 0 for player 2, because player 1 plays H, so player 2's best response is to play E
- But why would player 1 play H if they got to that choice node?
- The equilibrium relies on a "threat" from player 1 that is not **credible**.
- Subgame perfect equilibria are Nash equilibria that do not rely on non-credible threats.



Subgames

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The **subgame of G rooted at h** is the restriction of G to the descendants of h.

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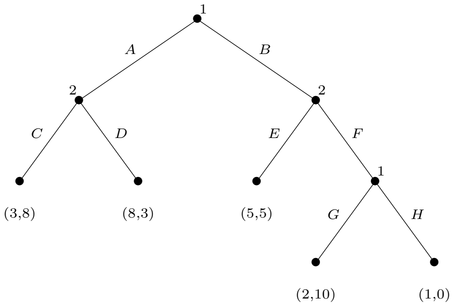

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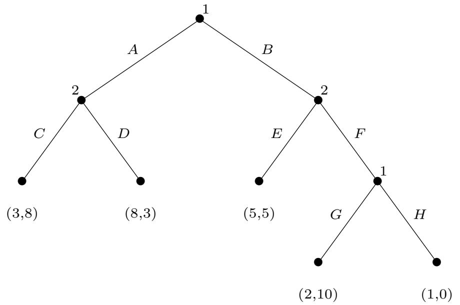

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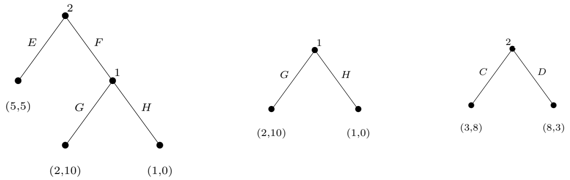

Subgame Perfect Equilibrium

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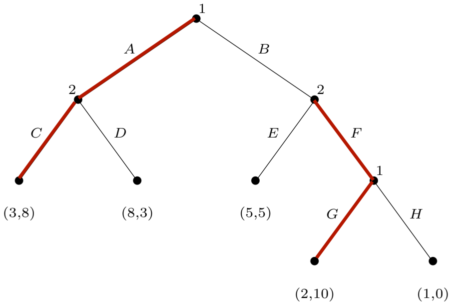

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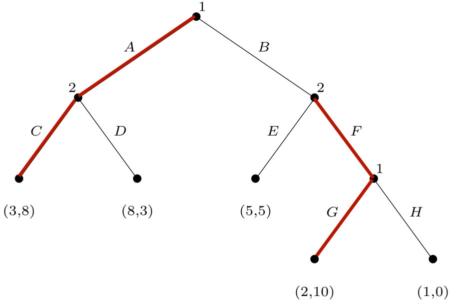

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Any equilibrium computed by backward induction will be subgame perfect (**Why?**)

Summary

- **Extensive form games** allow us to represent sequential action
  - Perfect information: when we see everything that happens
  - Imperfect information: different agents have different information
- Pure strategies for extensive form games map choice nodes to actions
  - Induced normal form is the normal form game with these pure strategies
  - Notions of mixed strategy, best response, etc. translate directly
- Subgame perfect equilibria are those which do not rely on non-credible threats
  - Can always find a subgame perfect equilibrium using backward induction
  - Furthermore, this equilibrium is guaranteed to be in pure strategies