Complexity of Nash Equilibrium

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(+ slides by Constantinos Daskalakis)

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Outline

1 Complexity Recap
2 Nash
3 Reduction from Nash
4 Reduction to Nash
Complexity Recap

Definition (P)
The set of decision problems that can be solved in polynomial time by a deterministic Turing machine. e.g., is this list sorted?

Definition (NP)
The set of decision problems that can be solved in polynomial time by a non-deterministic Turing machine. e.g., is this boolean formula satisfiable?
Definition (Reduction)

Transforming one problem into another (using a deterministic Turing machine).

\[ A \leq_P B \] means “Problem \( A \) can be solved using an algorithm for problem \( B \), with polynomial additional cost.”

- \( A \leq_P B \) and \( B \in NP \) implies \( A \in NP \).
Definition ($X$-hard)

A problem is $X$-hard iff it is at least as hard as any problem in $X$. 

- $A \leq_P B$ and $A$ is NP-hard implies $B$ is NP-hard.

Definition ($X$-complete)

A problem is $X$-complete iff it is in $X$ and $X$-hard.

- $A \leq_P B$, $B \leq_P A$ and $A$ is NP-complete implies $B$ is NP-complete.
Where does Nash fit in?

- As a decision problem, it’s easy:
  Does this game have a Nash equilibrium? Yes!
Where does Nash fit in?

- As a **decision problem**, it’s easy:
  Does this game have a Nash equilibrium? Yes!

- Ask slightly more and it becomes NP-complete, e.g.,
  - Does this game have more than one Nash equilibrium?
  - Does this game have a Nash equilibrium equilibrium where action $a_i$ is played with non-zero probability?
  - Does this game have a Nash equilibrium equilibrium where action $a_i$ is played with zero probability?

- But what’s the complexity of **finding** a Nash equilibrium?
Where does Nash fit in?

- What’s the complexity of finding a Nash equilibrium?

**Definition (FNP)**

The set of function problems that can be solved in polynomial time by a non-deterministic Turing machine. e.g., find a satisfying assignment for this boolean formula.

- $\epsilon$-NASH $\in$ FNP.
Where does Nash fit in?

- What’s the complexity of finding a Nash equilibrium?

**Definition (FNP)**

The set of function problems that can be solved in polynomial time by a non-deterministic Turing machine. e.g., find a satisfying assignment for this boolean formula.

- $\epsilon$-NASH $\in$ FNP.

- What’s that $\epsilon$ mean?
Where does Nash fit in?

- What’s the complexity of finding a Nash equilibrium?

**Definition (FNP)**

The set of function problems that can be solved in polynomial time by a non-deterministic Turing machine.

e.g., find a satisfying assignment for this boolean formula.

- $\epsilon$-NASH $\in$ FNP.
- What’s that $\epsilon$ mean?
- Where did the $\epsilon$ come from? Games with more than two players might not any rational-valued Nash equilibrium.
Where does Nash fit in?

**Definition (PPAD)**

The set of function problems where a solution is guaranteed to exist, by a parity argument on a directed graph.

- $\text{PPAD} \subseteq \text{FNP}$.

**Theorem (Daskalakis et al, Chen & Deng)**

$\epsilon$-Nash is PPAD-complete.
Where does Nash fit in?

**Definition (PPAD)**
The set of function problems where a solution is guaranteed to exist, by a parity argument on a directed graph.

- $\text{PPAD} \subseteq \text{FNP}$.

**Theorem (Daskalakis et al, Chen & Deng)**
$\epsilon$-Nash is PPAD-complete.

**Agenda:**
- Show $\epsilon$-NASH $\leq_P$ BROUWER (PPAD-complete)
  i.e., $\epsilon$-NASH $\in$ PPAD
- Show BROUWER $\leq_P$ $\epsilon$-NASH
  i.e., $\epsilon$-NASH is PPAD-hard.
Outline

1. Complexity Recap
2. Nash
3. Reduction from Nash
4. Reduction to Nash
Nash’s Theorem “⇒” Nash $\in$ PPAD

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<thead>
<tr>
<th>Kick</th>
<th>Left</th>
<th>Right</th>
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<tbody>
<tr>
<td>Dive</td>
<td>$1, -1$</td>
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<tr>
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Penalty Shot Game

$f: [0,1]^2 \rightarrow [0,1]^2$, cont. such that fixed point $\equiv$ Nash eq.
Nash’s Theorem “⇒” NASH ∈ PPAD

Nash

Penalty Shot Game

Brouwer

Kick

Dive

Left

Right

Penalty Shot Game

Pr[Right]
Nash’s Theorem “$\implies$” NASH $\in$ PPAD

Penalty Shot Game

<table>
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<td>Left</td>
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fixed point
Nash’s Theorem “⇒” NASH \in PPAD

Penalty Shot Game

\begin{array}{c|cc}
\text{Kick} & \frac{1}{2} & \frac{1}{2} \\
\hline
\text{Dive} & \text{Left} & \text{Right} \\
\hline
\text{Left} & 1, -1 & -1, 1 \\
\hline
\text{Right} & -1, 1 & 1, -1 \\
\end{array}

\epsilon\text{-fixed point}
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PPAD-Hardness of NASH [DGP ’05]

Nash \leftrightarrow Brouwer

- Game-gadgets: games acting as arithmetic gates

\textit{Nash} game whose Nash equilibria are close to the fixed points of \( f \)

\( f: [0,1]^3 \rightarrow [0,1]^3 \), continuous & p.w.linear
Games that do real arithmetic

Two strategies per player, say \{0,1\};

- Mixed strategy \equiv a number in [0,1] (probability of playing 1)

\[ w \text{ is paid:} \]
- $ p_x \cdot p_y \text{ for playing 0} \\
- $ p_z \text{ for playing 1} \\

\[ z \text{ is paid } 1 - p_w \text{ for playing 1} \]

\[ p_z = p_x \cdot p_y \]
Games that do real arithmetic

\begin{align*}
\text{for playing 0} \\
\text{for playing 1}
\end{align*}

<table>
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<th>y plays 0</th>
<th>y plays 1</th>
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<tbody>
<tr>
<td>x plays 0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>x plays 1</td>
<td>0</td>
<td>1</td>
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w's payoff

\begin{align*}
\text{w is paid:} \\
- \$ p_x \cdot p_y \text{ for playing 0} \\
- \$ p_z \text{ for playing 1}
\end{align*}

z is paid:

\begin{align*}
- \$1 - p_w \text{ for playing 1} \\
- \$0.5 \text{ for playing 0}
\end{align*}

p_z = p_x \cdot p_y
PPAD-Hardness of NASH [DGP ’05]

- use game-gadgets to simulate \( f \) with a game

- Topology: noise reduction

\[ f: [0,1]^3 \rightarrow [0,1]^3, \text{ continuous \\& p.w.linear} \]
Reduction to 3 players [Das, Pap ‘05]

multiplayer game
Reduction to 3 players [Das, Pap ‘05]

multiplayer game

― "represents" red players
― "represents" blue players
― "represents" all green players

Coloring: no two nodes affecting one another, or affecting the same third player use the same color;
payoffs of the green lawyer for representing node \( u \)

 Wishful thinking: The Nash equilibrium of the lawyer-game, gives a Nash equilibrium of the original multiplayer game, after marginalizing with respect to individual nodes.

But why would a lawyer represent every node equally?
Enforcing Fairness

律师们在一边玩一场高风险的游戏，他们代表的节点。
PPAD-hardness of NASH

Generic PPAD

Embedded PPAD

SPERNER

BROUWER

0^n

p.w. linear

BROUWER

multi-player NASH

[pap '94]

[DGP '05]

4-player NASH

[DGP '05]

3-player NASH

[DGP '05]

2-player NASH

[DGP '05]

[DGP '05]

[DP '05]

[CD'05]

[CD'05]
Reducing to 2 players [Chen, Deng ’05]

Based on the following simple, but crucial observation:
- the expected payoff of each lawyer is additive w.r.t. the nodes that another lawyer represents;
- hence, if two nodes affect the same third node, they don’t need to have different colors.

**Coloring:** no two nodes affecting one another, or affecting the same third player use the same color;

Two colors suffice to color the multiplayer game in the [DGP 05] construction.

2 lawyers are enough