Computing Domination; Correlated Equilibria

Lecture 6

Lecture Overview

1. Recap
2. Computational Problems Involving Domination
3. Rationalizability
4. Correlated Equilibrium
5. Computing Correlated Equilibria

Computing equilibria of zero-sum games

$$
\begin{aligned}
\text{minimize} \quad & U_1^* \\
\text{subject to} \quad & \sum_{a_2 \in A_2} u_1(a_1, a_2) \cdot s_2^{a_2} \leq U_1^* && \forall a_1 \in A_1 \\
& \sum_{a_2 \in A_2} s_2^{a_2} = 1 \\
& s_2^{a_2} \geq 0 && \forall a_2 \in A_2
\end{aligned}
$$

- This formulation gives us the minmax strategy for player 2.
- To get the minmax strategy for player 1, we need to solve a second (analogous) LP.

Computing Maxmin Strategies in General-Sum Games

To compute a maxmin strategy for player 1 in an arbitrary 2-player game G:

- Create a new game G' where player 2's payoffs are just the negatives of player 1's payoffs.
- By the minmax theorem, equilibrium strategies for player 1 in G' are equivalent to a maxmin strategies
- Thus, to find a maxmin strategy for G, find an equilibrium strategy for G'.

Domination

- Let s_i and s_i' be two strategies for player i, and let S_{-i} be is the set of all possible strategy profiles for the other players

Definition

s_i strictly dominates s_i' if $\forall s_{-i} \in S_{-i}$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$

Definition

s_i weakly dominates s_i' if $\forall s_{-i} \in S_{-i}$, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ and $\exists s_{-i} \in S_{-i}$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$

Definition

s_i very weakly dominates s_i' if $\forall s_{-i} \in S_{-i}$, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$

Iterated Removal of Dominated Strategies

- This process preserves Nash equilibria.
  - strict dominance: all equilibria preserved.
  - weak or very weak dominance: at least one equilibrium preserved.
- Thus, it can be used as a preprocessing step before computing an equilibrium
  - Some games are solvable using this technique.
- What about the order of removal when there are multiple dominated strategies?
  - strict dominance: doesn't matter.
  - weak or very weak dominance: can affect which equilibria are preserved.

Is s_i strictly dominated by any pure strategy?

Try to identify some pure strategy that is strictly better than s_i for any pure strategy profile of the others.

for all pure strategies $a_i \in A_i$ for player i where $a_i \neq s_i$ **do**
  $dom \leftarrow true$
  for all pure strategy profiles $a_{-i} \in A_{-i}$ for the players other than i **do**
    if $u_i(s_i, a_{-i}) \geq u_i(a_i, a_{-i})$ **then**
      $dom \leftarrow false$
      break
    end if
  end for
  if $dom = true$ **then return** $true$
end for
return $false$

Lecture Overview

Computational Problems in Domination

- Identifying strategies dominated by a pure strategy (done)
- Identifying strategies dominated by a mixed strategy
- Identifying strategies that survive iterated elimination
- Asking whether a strategy survives iterated elimination under all elimination orderings
- We'll assume that i's utility function is strictly positive everywhere (why is this OK?)

Constraints for determining whether s_i is strictly dominated by any mixed strategy

$$\sum_{j \in A_i} p_j u_i(a_j, a_{-i}) > u_i(s_i, a_{-i}) \qquad \forall a_{-i} \in A_{-i}$$

$$p_j \geq 0 \qquad \forall j \in A_i$$

$$\sum_{j \in A_i} p_j = 1$$

Constraints for determining whether s_i is strictly dominated by any mixed strategy

$$\sum_{j \in A_i} p_j u_i(a_j, a_{-i}) > u_i(s_i, a_{-i}) \qquad \forall a_{-i} \in A_{-i}$$

$$p_j \geq 0 \qquad \forall j \in A_i$$

$$\sum_{j \in A_i} p_j = 1$$

- What's wrong with this program?

Constraints for determining whether s_i is strictly dominated by any mixed strategy

$$\sum_{j \in A_i} p_j u_i(a_j, a_{-i}) > u_i(s_i, a_{-i}) \qquad \forall a_{-i} \in A_{-i}$$

$$p_j \geq 0 \qquad \forall j \in A_i$$

$$\sum_{j \in A_i} p_j = 1$$

- What's wrong with this program?
  - strict inequality in the first constraint means we don't have an LP

LP for determining whether s_i is strictly dominated by any mixed strategy

$$
\begin{aligned}
\text{minimize} \quad & \sum_{j \in A_i} p_j \\
\text{subject to} \quad & \sum_{j \in A_i} p_j u_i(a_j, a_{-i}) \geq u_i(s_i, a_{-i}) && \forall a_{-i} \in A_{-i} \\
& p_j \geq 0 && \forall j \in A_i
\end{aligned}
$$

- This is clearly an LP. Why is it a solution to our problem?

LP for determining whether s_i is strictly dominated by any mixed strategy

$$\begin{aligned} \text{minimize} \quad & \sum_{j \in A_i} p_j \\ \text{subject to} \quad & \sum_{j \in A_i} p_j u_i(a_j, a_{-i}) \geq u_i(s_i, a_{-i}) && \forall a_{-i} \in A_{-i} \\ & p_j \geq 0 && \forall j \in A_i \end{aligned}$$

- This is clearly an LP. Why is it a solution to our problem?
  - if a solution exists with $\sum_j p_j < 1$ then we can add $1 - \sum_j p_j$ to some p_k and we'll have a dominating mixed strategy (since utility was assumed to be positive everywhere)
- Our original approach works for very weak domination
- For weak domination we can use that program with a different objective function trick.

Identifying strategies that survive iterated elimination

- This can be done by repeatedly solving our LPs: solving a polynomial number of LPs is still in $\mathcal{P}$.
  - Checking whether every pure strategy of every player is dominated by any other mixed strategy requires us to solve at worst $\sum_{i \in N} |A_i|$ linear programs.
  - Each step removes one pure strategy for one player, so there can be at most $\sum_{i \in N}(|A_i| - 1)$ steps.
  - Thus we need to solve $O((n \cdot \max_i |A_i|)^2)$ linear programs.

Further questions about iterated elimination

1. **(Strategy Elimination)** Does there exist some elimination path under which the strategy s_i is eliminated?
2. **(Reduction Identity)** Given action subsets $A'_i \subseteq A_i$ for each player i, does there exist a maximally reduced game where each player i has the actions A'_i?
3. **(Uniqueness)** Does every elimination path lead to the same reduced game?
4. **(Reduction Size)** Given constants k_i for each player i, does there exist a maximally reduced game where each player i has exactly k_i actions?

Further questions about iterated elimination

1. **(Strategy Elimination)** Does there exist some elimination path under which the strategy s_i is eliminated?
2. **(Reduction Identity)** Given action subsets $A'_i \subseteq A_i$ for each player i, does there exist a maximally reduced game where each player i has the actions A'_i?
3. **(Uniqueness)** Does every elimination path lead to the same reduced game?
4. **(Reduction Size)** Given constants k_i for each player i, does there exist a maximally reduced game where each player i has exactly k_i actions?

- For iterated strict dominance these problems are all in $\mathcal{P}$.
- For iterated weak or very weak dominance these problems are all $\mathcal{NP}$-complete.

Lecture Overview

Rationalizability

- Rather than ask what is irrational, ask what is a best response to some beliefs about the opponent
  - assumes opponent is rational
  - assumes opponent knows that you and the others are rational
  - ...
- Examples
  - is *heads* rational in matching pennies?

Rationalizability

- Rather than ask what is irrational, ask what is a best response to some beliefs about the opponent
  - assumes opponent is rational
  - assumes opponent knows that you and the others are rational
  - ...
- Examples
  - is $heads$ rational in matching pennies?
  - is $cooperate$ rational in prisoner's dilemma?

Rationalizability

- Rather than ask what is irrational, ask what is a best response to some beliefs about the opponent
  - assumes opponent is rational
  - assumes opponent knows that you and the others are rational
  - ...
- Examples
  - is *heads* rational in matching pennies?
  - is *cooperate* rational in prisoner's dilemma?
- Will there always exist a rationalizable strategy?

Rationalizability

- Rather than ask what is irrational, ask what is a best response to some beliefs about the opponent
  - assumes opponent is rational
  - assumes opponent knows that you and the others are rational
  - ...
- Examples
  - is $heads$ rational in matching pennies?
  - is $cooperate$ rational in prisoner's dilemma?
- Will there always exist a rationalizable strategy?
  - Yes, equilibrium strategies are always rationalizable.

Rationalizability

- Rather than ask what is irrational, ask what is a best response to some beliefs about the opponent
  - assumes opponent is rational
  - assumes opponent knows that you and the others are rational
  - ...
- Examples
  - is $heads$ rational in matching pennies?
  - is $cooperate$ rational in prisoner's dilemma?
- Will there always exist a rationalizable strategy?
  - Yes, equilibrium strategies are always rationalizable.
- Furthermore, in two-player games, rationalizable $\Leftrightarrow$ survives iterated removal of strictly dominated strategies.

Lecture Overview

Pithy Quote

If there is intelligent life on other planets, in a majority of them, they would have discovered correlated equilibrium before Nash equilibrium.

– Roger Myerson

Examples

- Consider again Battle of the Sexes.
  - Intuitively, the best outcome seems a 50-50 split between (F, F) and (B, B).
  - But there's no way to achieve this, so either someone loses out (unfair) or both players often miscoordinate
- Another classic example: traffic game

	go	$wait$
go	$-100, -100$	$10, 0$
B	$0, 10$	$-10, -10$

Intuition

- What is the natural solution here?

Intuition

- What is the natural solution here?
  - A traffic light: a fair randomizing device that tells one of the agents to go and the other to wait.
- Benefits:
  - the negative payoff outcomes are completely avoided
  - fairness is achieved
  - the sum of social welfare exceeds that of any Nash equilibrium
- We could use the same idea to achieve the fair outcome in battle of the sexes.
- Our example presumed that everyone perfectly observes the random event; not required.
- More generally, some random variable with a commonly known distribution, and a private signal to each player about the outcome.
  - signal doesn't determine the outcome or others' signals; however, correlated

Formal definition

Definition (Correlated equilibrium)

Given an n-agent game $G = (N, A, u)$, a correlated equilibrium is a tuple (v, π, σ), where v is a tuple of random variables $v = (v_1, \ldots, v_n)$ with respective domains $D = (D_1, \ldots, D_n)$, π is a joint distribution over v, $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a vector of mappings $\sigma_i : D_i \mapsto A_i$, and for each agent i and every mapping $\sigma_i' : D_i \mapsto A_i$ it is the case that

$$\sum_{d \in D} \pi(d) u_i \left(\sigma_1(d_1), \ldots, \sigma_i(d_i), \ldots, \sigma_n(d_n)\right) \geq \sum_{d \in D} \pi(d) u_i \left(\sigma_1(d_1), \ldots, \sigma_i'(d_i), \ldots, \sigma_n(d_n)\right).$$

Existence

Theorem

For every Nash equilibrium σ^* there exists a corresponding correlated equilibrium σ.

- This is easy to show:
  - let $D_i = A_i$
  - let $\pi(d) = \prod_{i \in N} \sigma_i^*(d_i)$
  - σ_i maps each d_i to the corresponding a_i.
- Thus, correlated equilibria always exist

Remarks

- Not every correlated equilibrium is equivalent to a Nash equilibrium
  - thus, correlated equilibrium is a weaker notion than Nash
- Any convex combination of the payoffs achievable under correlated equilibria is itself realizable under a correlated equilibrium
  - start with the Nash equilibria (each of which is a CE)
  - introduce a second randomizing device that selects which CE the agents will play
  - regardless of the probabilities, no agent has incentive to deviate
  - the probabilities can be adjusted to achieve any convex combination of the equilibrium payoffs
  - the randomizing devices can be combined

Lecture Overview

Computing CE

$$\sum_{a \in A | a_i \in a} p(a) u_i(a) \geq \sum_{a \in A | a_i \in a} p(a) u_i(a_i', a_{-i}) \quad \forall i \in N,\ \forall a_i, a_i' \in A_i$$

$$p(a) \geq 0 \quad \forall a \in A$$

$$\sum_{a \in A} p(a) = 1$$

- variables: $p(a)$; constants: $u_i(a)$

Computing CE

$$\sum_{a \in A | a_i \in a} p(a) u_i(a) \geq \sum_{a \in A | a_i \in a} p(a) u_i(a'_i, a_{-i}) \quad \forall i \in N, \forall a_i, a'_i \in A_i$$

$$p(a) \geq 0 \quad \forall a \in A$$

$$\sum_{a \in A} p(a) = 1$$

- variables: $p(a)$; constants: $u_i(a)$
- we could find the social-welfare maximizing CE by adding an objective function

$$\text{maximize:} \quad \sum_{a \in A} p(a) \sum_{i \in N} u_i(a).$$

Why are CE easier to compute than NE?

$$\sum_{a \in A | a_i \in a} p(a) u_i(a) \geq \sum_{a \in A | a'_i \in a} p(a) u_i(a'_i, a_{-i}) \quad \forall i \in N, \forall a_i, a'_i \in A_i$$

$$p(a) \geq 0 \quad \forall a \in A$$

$$\sum_{a \in A} p(a) = 1$$

- intuitively, correlated equilibrium has only a single randomization over outcomes, whereas in NE this is constructed as a product of independent probabilities.
- To change this program so that it finds NE, the first constraint would be

$$\sum_{a \in A} u_i(a) \prod_{j \in N} p_j(a_j) \geq \sum_{a \in A} u_i(a'_i, a_{-i}) \prod_{j \in N \setminus \{i\}} p_j(a_j) \quad \forall i \in N, \forall a'_i \in A_i.$$

- This is a nonlinear constraint!