Computing Minmax; Dominance

CPSC 532A Lecture 5

Lecture Overview

1. Recap
2. Linear Programming
3. Computational Problems Involving Maxmin
4. Domination
5. Fun Game
6. Iterated Removal of Dominated Strategies
7. Computational Problems Involving Domination

What are solution concepts?

- Solution concept: a subset of the outcomes in the game that are somehow interesting.
- There is an implicit computational problem of finding these outcomes given a particular game.
- Depending on the concept, existence can be an issue.

Solution concepts we've seen so far:

What are solution concepts?

- Solution concept: a subset of the outcomes in the game that are somehow interesting.
- There is an implicit computational problem of finding these outcomes given a particular game.
- Depending on the concept, existence can be an issue.

Solution concepts we've seen so far:

- Pareto-optimal outcome
- Pure-strategy Nash equilibrium
- Mixed-strategy Nash equilibrium
- Other Nash variants:
  - weak Nash equilibrium
  - strict Nash equilibrium
- maxmin strategy profile
- minmax strategy profile

Maxmin and Minmax

Definition (Maxmin)

The maxmin strategy for player i is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_1, s_2)$, and the maxmin value for player i is $\max_{s_i} \min_{s_{-i}} u_i(s_1, s_2)$.

Definition (Minmax, 2-player)

In a two-player game, the minmax strategy for player i against player $-i$ is $\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$, and player $-i$'s minmax value is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$.

We can also generalize minmax to n players.

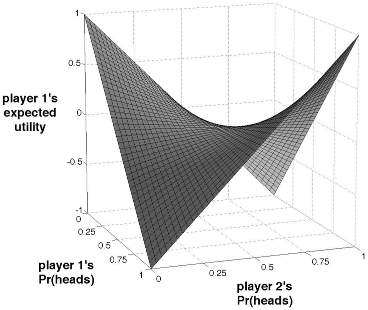
Minmax Theorem

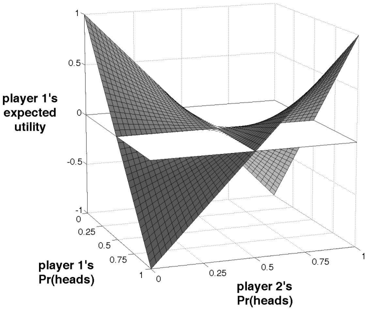
Theorem (Minimax theorem (von Neumann, 1928))

In any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.

1. Each player's maxmin value is equal to his minmax value. By convention, the maxmin value for player 1 is called the value of the game.
2. For both players, the set of maxmin strategies coincides with the set of minmax strategies.
3. Any maxmin strategy profile (or, equivalently, minmax strategy profile) is a Nash equilibrium. Furthermore, these are all the Nash equilibria. Consequently, all Nash equilibria have the same payoff vector (namely, those in which player 1 gets the value of the game).

Saddle Point: Matching Pennies





Lecture Overview

Linear Programming

A linear program is defined by:

- a set of real-valued variables
- a linear objective function
  - a weighted sum of the variables
- a set of linear constraints
  - the requirement that a weighted sum of the variables must be greater than or equal to some constant

Linear Programming

Given n variables and m constraints, variables x and constants w, a and b:

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} w_i x_i \\ \text{subject to} \quad & \sum_{i=1}^{n} a_{ij} x_i \leq b_j \qquad && \forall j = 1 \ldots m \\ & x_i \in \{0, 1\} && \forall i = 1 \ldots n \end{aligned}$$

- These problems can be solved in polynomial time using interior point methods.
  - Interestingly, the (worst-case exponential) simplex method is often faster in practice.

Lecture Overview

Computing equilibria of zero-sum games

$$
\begin{aligned}
\text{minimize} \quad & U_1^* \\
\text{subject to} \quad & \sum_{a_2 \in A_2} u_1(a_1, a_2) \cdot s_2^{a_2} \leq U_1^* && \forall a_1 \in A_1 \\
& \sum_{a_2 \in A_2} s_2^{a_2} = 1 \\
& s_2^{a_2} \geq 0 && \forall a_2 \in A_2
\end{aligned}
$$

- First, identify the variables:
  - U_1^* is the expected utility for player 1
  - $s_2^{a_2}$ is player 2's probability of playing action a_2 under his mixed strategy
- each $u_1(a_1, a_2)$ is a constant.

Computing equilibria of zero-sum games

Now let's interpret the LP:

$$
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$$

- s_2 is a valid probability distribution.

Computing equilibria of zero-sum games

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- U_1^* is as small as possible.

Computing equilibria of zero-sum games

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& \sum_{a_2 \in A_2} s_2^{a_2} = 1 \\
& s_2^{a_2} \geq 0 && \forall a_2 \in A_2
\end{aligned}
$$

- Player 1's expected utility for playing each of his actions under player 2's mixed strategy is no more than U_1^*.
  - Because U_1^* is minimized, this constraint will be tight for some actions: the support of player 1's mixed strategy.

Computing equilibria of zero-sum games

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\begin{aligned}
\text{minimize} \quad & U_1^* \\
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& \sum_{a_2 \in A_2} s_2^{a_2} = 1 \\
& s_2^{a_2} \geq 0 & \forall a_2 \in A_2
\end{aligned}
$$

- This formulation gives us the minmax strategy for player 2.
- To get the minmax strategy for player 1, we need to solve a second (analogous) LP.

Computing Maxmin Strategies in General-Sum Games

Let's say we want to compute a maxmin strategy for player 1 in an arbitrary 2-player game G.

Computing Maxmin Strategies in General-Sum Games

Let's say we want to compute a maxmin strategy for player 1 in an arbitrary 2-player game G.

- Create a new game G' where player 2's payoffs are just the negatives of player 1's payoffs.
- The maxmin strategy for player 1 in G does not depend on player 2's payoffs
  - Thus, the maxmin strategy for player 1 in G is the same as the maxmin strategy for player 1 in G'
- By the minmax theorem, equilibrium strategies for player 1 in G' are equivalent to a maxmin strategies
- Thus, to find a maxmin strategy for G, find an equilibrium strategy for G'.

Lecture Overview

Domination

- Let s_i and s'_i be two strategies for player i, and let S_{-i} be is the set of all possible strategy profiles for the other players

Definition

s_i strictly dominates s'_i if $\forall s_{-i} \in S_{-i},\ u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$

Definition

s_i weakly dominates s'_i if $\forall s_{-i} \in S_{-i},\ u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ and $\exists s_{-i} \in S_{-i},\ u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$

Definition

s_i very weakly dominates s'_i if $\forall s_{-i} \in S_{-i},\ u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$

Equilibria and dominance

- If one strategy dominates all others, we say it is dominant.
- A strategy profile consisting of dominant strategies for every player must be a Nash equilibrium.
  - An equilibrium in strictly dominant strategies must be unique.

Equilibria and dominance

- If one strategy dominates all others, we say it is dominant.
- A strategy profile consisting of dominant strategies for every player must be a Nash equilibrium.
  - An equilibrium in strictly dominant strategies must be unique.
- Consider Prisoner's Dilemma again
  - not only is the only equilibrium the only non-Pareto-optimal outcome, but it's also an equilibrium in strictly dominant strategies!

Lecture Overview

Traveler's Dilemma

Two travelers purchase identical African masks while on a tropical vacation. Their luggage is lost on the return trip, and the airline asks them to make independent claims for compensation. In anticipation of excessive claims, the airline representative announces: "We know that the bags have identical contents, and we will entertain any claim between \$180 and \$300, but you will each be reimbursed at an amount that equals the minimum of the two claims submitted. If the two claims differ, we will also pay a reward R *to the person making the smaller claim and we will deduct a penalty* R *from the reimbursement to the person making the larger claim."*

Traveler's Dilemma

- Action: choose an integer between 180 and 300
- If both players pick the same number, they both get that amount as payoff
- If players pick a different number:
  - the low player gets his number (L) plus some constant R
  - the high player gets $L - R$, $R = 5$.
- Play this game *once* with a partner; play with as many different partners as you like.

Traveler's Dilemma

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- If both players pick the same number, they both get that amount as payoff
- If players pick a different number:
  - the low player gets his number (L) plus some constant R
  - the high player gets $L - R$, $R = 5$.
- Play this game *once* with a partner; play with as many different partners as you like.
  - Now set $R = 180$, and again play with as many partners as you like.

Traveler's Dilemma

- What is the equilibrium?

Traveler's Dilemma

- What is the equilibrium?
  - $(180, 180)$ is the only equilibrium, for all $R \geq 2$.

Traveler's Dilemma

- What is the equilibrium?
  - $(180, 180)$ is the only equilibrium, for all $R \geq 2$.
- What happens?

Traveler's Dilemma

- What is the equilibrium?
  - $(180, 180)$ is the only equilibrium, for all $R \geq 2$.
- What happens?
  - with $R = 5$ most people choose 295–300
  - with $R = 180$ most people choose 180

Lecture Overview

Dominated strategies

- No equilibrium can involve a strictly dominated strategy
  - Thus we can remove it, and end up with a strategically equivalent game
  - This might allow us to remove another strategy that wasn't dominated before
  - Running this process to termination is called iterated removal of dominated strategies.

Iterated Removal of Dominated Strategies: Example

	L	C	R
U	$3, 1$	$0, 1$	$0, 0$
M	$1, 1$	$1, 1$	$5, 0$
D	$0, 1$	$4, 1$	$0, 0$

Iterated Removal of Dominated Strategies: Example

	L	C	R
U	$3, 1$	$0, 1$	$0, 0$
M	$1, 1$	$1, 1$	$5, 0$
D	$0, 1$	$4, 1$	$0, 0$

- R is dominated by L.

Iterated Removal of Dominated Strategies: Example

	L	C
U	$3, 1$	$0, 1$
M	$1, 1$	$1, 1$
D	$0, 1$	$4, 1$

Iterated Removal of Dominated Strategies: Example

	L	C
U	$3, 1$	$0, 1$
M	$1, 1$	$1, 1$
D	$0, 1$	$4, 1$

- M is dominated by the mixed strategy that selects U and D with equal probability.

Iterated Removal of Dominated Strategies: Example

	L	C
U	$3, 1$	$0, 1$
D	$0, 1$	$4, 1$

Iterated Removal of Dominated Strategies: Example

	L	C
U	$3, 1$	$0, 1$
D	$0, 1$	$4, 1$

- No other strategies are dominated.

Iterated Removal of Dominated Strategies

- This process preserves Nash equilibria.
  - strict dominance: all equilibria preserved.
  - weak or very weak dominance: at least one equilibrium preserved.
- Thus, it can be used as a preprocessing step before computing an equilibrium
  - Some games are solvable using this technique.
  - Example: Traveler's Dilemma!
- What about the order of removal when there are multiple dominated strategies?
  - strict dominance: doesn't matter.
  - weak or very weak dominance: can affect which equilibria are preserved.

Lecture Overview

Computational Problems in Domination

- Identifying strategies dominated by a pure strategy
- Identifying strategies dominated by a mixed strategy
- Identifying strategies that survive iterated elimination
- Asking whether a strategy survives iterated elimination under all elimination orderings
- We'll assume that i's utility function is strictly positive everywhere (why is this OK?)

Is s_i strictly dominated by any pure strategy?

Try to identify some pure strategy that is strictly better than s_i for any pure strategy profile of the others.

```
for all pure strategies a_i ∈ A_i for player i where a_i ≠ s_i do
    dom ← true
    for all pure strategy profiles a_{-i} ∈ A_{-i} for the players other than i do
        if u_i(s_i, a_{-i}) ≥ u_i(a_i, a_{-i}) then
            dom ← false
            break
        end if
    end for
    if dom = true then return true
end for
return false
```

Is s_i strictly dominated by any pure strategy?

Try to identify some pure strategy that is strictly better than s_i for any pure strategy profile of the others.

for all pure strategies $a_i \in A_i$ for player i where $a_i \neq s_i$ **do**
  $dom \leftarrow true$
  for all pure strategy profiles $a_{-i} \in A_{-i}$ for the players other than i **do**
    if $u_i(s_i, a_{-i}) \geq u_i(a_i, a_{-i})$ **then**
      $dom \leftarrow false$
      break
    end if
  end for
  if $dom = true$ **then return** $true$
end for
return $false$

- What is the complexity of this procedure?
- Why don't we have to check mixed strategies of $-i$?
- Minor changes needed to test for weak, very weak dominance.