Computing Nash Equilibrium; Maxmin

Lecture 5

Computing Nash Equilibrium; Maxmin

Lecture 5, Slide 1

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Lecture Overview



2 Computing Mixed Nash Equilibria





Computing Nash Equilibrium; Maxmin



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- Idea: sometimes, one outcome o is at least as good for every agent as another outcome o^\prime , and there is some agent who strictly prefers o to o^\prime
 - ${\, \bullet \,}$ in this case, it seems reasonable to say that o is better than o'
 - we say that *o* Pareto-dominates *o*'.

- An outcome o^* is Pareto-optimal if there is no other outcome that Pareto-dominates it.
 - a game can have more than one Pareto-optimal outcome
 - every game has at least one Pareto-optimal outcome

• If you knew what everyone else was going to do, it would be easy to pick your own action

• Let
$$a_{-i} = \langle a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \rangle$$
.

• now
$$a = (a_{-i}, a_i)$$

• Best response: $a_i^* \in BR(a_{-i})$ iff $\forall a_i \in A_i, u_i(a_i^*, a_{-i}) \ge u_i(a_i, a_{-i})$ • Now we return to the setting where no agent knows anything about what the others will do

- Idea: look for stable action profiles.
- $a = \langle a_1, \ldots, a_n \rangle$ is a ("pure strategy") Nash equilibrium iff $\forall i, a_i \in BR(a_{-i})$.

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Recap	Computing Mixed NE	Fun Game	Maxmin and Minmax
Mixed S	Strategies		

- It would be a pretty bad idea to play any deterministic strategy in matching pennies
- Idea: confuse the opponent by playing randomly
- Define a strategy s_i for agent i as any probability distribution over the actions A_i .
 - pure strategy: only one action is played with positive probability
 - mixed strategy: more than one action is played with positive probability
 - these actions are called the support of the mixed strategy
- Let the set of all strategies for i be S_i
- Let the set of all strategy profiles be $S = S_1 \times \ldots \times S_n$.

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Utility under Mixed Strategies

- What is your payoff if all the players follow mixed strategy profile $s \in S$?
 - We can't just read this number from the game matrix anymore: we won't always end up in the same cell
- Instead, use the idea of expected utility from decision theory:

$$u_i(s) = \sum_{a \in A} u_i(a) Pr(a|s)$$
$$Pr(a|s) = \prod_{j \in N} s_j(a_j)$$

Computing Nash Equilibrium; Maxmin

Best Response and Nash Equilibrium

Our definitions of best response and Nash equilibrium generalize from actions to strategies.

- Best response:
 - $s_i^* \in BR(s_{-i})$ iff $\forall s_i \in S_i, u_i(s_i^*, s_{-i}) \ge u_i(s_i, s_{-i})$
- Nash equilibrium:
 - $s = \langle s_1, \ldots, s_n \rangle$ is a Nash equilibrium iff $\forall i, s_i \in BR(s_{-i})$
- Every finite game has a Nash equilibrium! [Nash, 1950]
 e.g., matching pennies: both players play heads/tails 50%/50%

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Lecture Overview



2 Computing Mixed Nash Equilibria





Computing Nash Equilibrium; Maxmin

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В	2, 1	0,0
F	0,0	1, 2

- It's hard in general to compute Nash equilibria, but it's easy when you can guess the support
- For BoS, let's look for an equilibrium where all actions are part of the support



- Let player 2 play B with p, F with 1 p.
- If player 1 best-responds with a mixed strategy, player 2 must make him indifferent between F and B (why?)



- Let player 2 play B with p, F with 1-p.
- If player 1 best-responds with a mixed strategy, player 2 must make him indifferent between F and B (why?)

$$u_1(B) = u_1(F)$$

 $2p + 0(1-p) = 0p + 1(1-p)$
 $p = \frac{1}{3}$



- Likewise, player 1 must randomize to make player 2 indifferent.
 - Why is player 1 willing to randomize?



- Likewise, player 1 must randomize to make player 2 indifferent.
 - Why is player 1 willing to randomize?
- Let player 1 play B with q, F with 1-q.

$$u_2(B) = u_2(F)$$

 $q + 0(1 - q) = 0q + 2(1 - q)$
 $q = \frac{2}{3}$

• Thus the strategies $(\frac{2}{3}, \frac{1}{3})$, $(\frac{1}{3}, \frac{2}{3})$ are a Nash equilibrium.

Interpreting Mixed Strategy Equilibria

What does it mean to play a mixed strategy?

- Randomize to confuse your opponent
 - consider the matching pennies example
- Players randomize when they are uncertain about the other's action
 - consider battle of the sexes
- Mixed strategies are a concise description of what might happen in repeated play: count of pure strategies in the limit
- Mixed strategies describe population dynamics: 2 agents chosen from a population, all having deterministic strategies. MS is the probability of getting an agent who will play one PS or another.

Lecture Overview



2 Computing Mixed Nash Equilibria



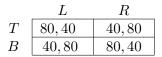


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Recap	Computing Mixed NE	Fun Game	Maxmin and Minmax
Fun Game!			



• Play once as each player, recording the strategy you follow.

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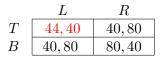
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- Play once as each player, recording the strategy you follow.
- What does row player do in equilibrium of this game?
 - row player randomizes 50-50 all the time
 - that's what it takes to make column player indifferent
- What happens when people play this game?
 - with payoff of 320, row player goes up essentially all the time
 - with payoff of 44, row player goes down essentially all the time

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2 Computing Mixed Nash Equilibria





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Maxmin Strategies

- Player *i*'s maxmin strategy is a strategy that maximizes *i*'s worst-case payoff, in the situation where all the other players (whom we denote -i) happen to play the strategies which cause the greatest harm to *i*.
- The maxmin value (or safety level) of the game for player *i* is that minimum amount of payoff guaranteed by a maxmin strategy.

Definition (Maxmin)

The maxmin strategy for player *i* is $\arg \max_{s_i} \min_{s_{-i}} u_i(s_1, s_2)$, and the maxmin value for player *i* is $\max_{s_i} \min_{s_{-i}} u_i(s_1, s_2)$.

• Why would *i* want to play a maxmin strategy?

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- Why would *i* want to play a maxmin strategy?
 - a conservative agent maximizing worst-case payoff
 - a paranoid agent who believes everyone is out to get him

Minmax Strategies

- Player *i*'s minmax strategy against player -i in a 2-player game is a strategy that minimizes -i's best-case payoff, and the minmax value for *i* against -i is payoff.
- Why would *i* want to play a minmax strategy?

Definition (Minmax, 2-player)

In a two-player game, the minmax strategy for player i against player -i is $\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$, and player -i's minmax value is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$.

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We can generalize to n players.

Definition (Minmax, *n*-player)

In an *n*-player game, the minmax strategy for player i against player $j \neq i$ is i's component of the mixed strategy profile s_{-j} in the expression $\arg\min_{s_{-j}}\max_{s_j}u_j(s_j,s_{-j})$, where -j denotes the set of players other than j. As before, the minmax value for player j is $\min_{s_{-j}}\max_{s_j}u_j(s_j,s_{-j})$.

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Theorem (Minimax theorem (von Neumann, 1928))

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- For both players, the set of maxmin strategies coincides with the set of minmax strategies.

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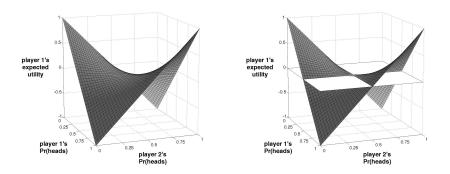
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- Each player's maxmin value is equal to his minmax value. By convention, the maxmin value for player 1 is called the value of the game.
- For both players, the set of maxmin strategies coincides with the set of minmax strategies.
- Any maxmin strategy profile (or, equivalently, minmax strategy profile) is a Nash equilibrium. Furthermore, these are all the Nash equilibria. Consequently, all Nash equilibria have the same payoff vector (namely, those in which player 1 gets the value of the game).

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Maxmin and Minmax

Saddle Point: Matching Pennies



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