From Optimality to Equilibrium

Lecture 4

From Optimality to Equilibrium

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Lecture Overview



2 Pareto Optimality

3 Best Response and Nash Equilibrium

4 Mixed Strategies

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Non-Cooperative Game Theory

• What is it?

Recap

• mathematical study of interaction between rational, self-interested agents

- Why is it called non-cooperative?
 - while it's most interested in situations where agents' interests conflict, it's not restricted to these settings
 - the key is that the individual is the basic modeling unit, and that individuals pursue their own interests
 - cooperative/coalitional game theory has teams as the central unit, rather than agents

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 Recap
 Pareto Optimality
 Best Response and Nash Equilibrium
 Mixed Strategies

 Defining Games
 Image: Compared Strategies
 Image: Compared Strategies
 Image: Compared Strategies

- Finite, *n*-person game: $\langle N, A, u \rangle$:
 - N is a finite set of n players, indexed by i
 - $A = A_1 \times \ldots \times A_n$, where A_i is the action set for player i
 - $a \in A$ is an action profile, and so A is the space of action profiles
 - $u = \langle u_1, \dots, u_n \rangle$, a utility function for each player, where $u_i : A \mapsto \mathbb{R}$
- Writing a 2-player game as a matrix:
 - row player is player 1, column player is player 2
 - rows are actions $a \in A_1$, columns are $a' \in A_2$
 - cells are outcomes, written as a tuple of utility values for each player

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Prisoner's dilemma

Prisoner's dilemma is any game

 $\begin{array}{c|c} C & D \\ \hline \\ C & a, a & b, c \\ \hline \\ D & c, b & d, d \end{array}$

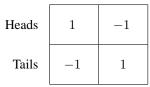
with c > a > d > b.

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Games of Pure Competition

Players have exactly opposed interests

- There must be precisely two players (otherwise they can't have exactly opposed interests)
- For all action profiles $a \in A$, $u_1(a) + u_2(a) = c$ for some constant c
 - Special case: zero sum



Heads Tails

Right

Games of Cooperation

Players have exactly the same interests.

• no conflict: all players want the same things

•
$$\forall a \in A, \forall i, j, u_i(a) = u_j(a)$$

Left	1	0
Right	0	1

Left

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General Games: Battle of the Sexes

The most interesting games combine elements of cooperation *and* competition.

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- We've defined some canonical games, and thought about how to play them. Now let's examine the games from the outside
- From the point of view of an outside observer, can some outcomes of a game be said to be better than others?

Analyzing Games

- We've defined some canonical games, and thought about how to play them. Now let's examine the games from the outside
- From the point of view of an outside observer, can some outcomes of a game be said to be better than others?
 - we have no way of saying that one agent's interests are more important than another's
 - intuition: imagine trying to find the revenue-maximizing outcome when you don't know what currency has been used to express each agent's payoff
- Are there situations where we can still prefer one outcome to another?

- Idea: sometimes, one outcome o is at least as good for every agent as another outcome o', and there is some agent who strictly prefers o to o'
 - $\bullet\,$ in this case, it seems reasonable to say that o is better than o'
 - we say that *o* Pareto-dominates *o*'.

- Idea: sometimes, one outcome o is at least as good for every agent as another outcome o^\prime , and there is some agent who strictly prefers o to o^\prime
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• An outcome o^* is Pareto-optimal if there is no other outcome that Pareto-dominates it.

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 - can a game have more than one Pareto-optimal outcome?

Pareto Optimality

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 - in this case, it seems reasonable to say that o is better than o'
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- An outcome o^{*} is Pareto-optimal if there is no other outcome that Pareto-dominates it.
 - can a game have more than one Pareto-optimal outcome?
 - does every game have at least one Pareto-optimal outcome?

Pareto Optimal Outcomes in Example Games

$$C$$
 D

$$\begin{array}{c|c|c} C & -1, -1 & -4, 0 \\ \hline D & 0, -4 & -3, -3 \end{array}$$

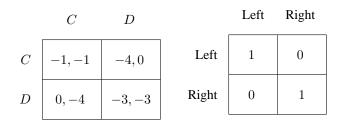
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Pareto Optimal Outcomes in Example Games



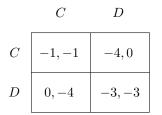
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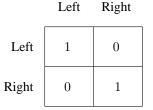
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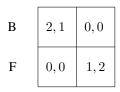
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Pareto Optimal Outcomes in Example Games





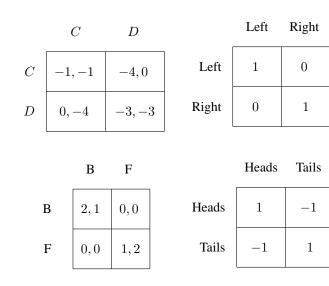
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Pareto Optimal Outcomes in Example Games



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3 Best Response and Nash Equilibrium

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• If you knew what everyone else was going to do, it would be easy to pick your own action

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• If you knew what everyone else was going to do, it would be easy to pick your own action

• Let
$$a_{-i} = \langle a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \rangle$$
.

• now
$$a = (a_{-i}, a_i)$$

• Best response: $a_i^* \in BR(a_{-i})$ iff $\forall a_i \in A_i, u_i(a_i^*, a_{-i}) \ge u_i(a_i, a_{-i})$

- Now let's return to the setting where no agent knows anything about what the others will do
- What can we say about which actions will occur?

- Now let's return to the setting where no agent knows anything about what the others will do
- What can we say about which actions will occur?

- Idea: look for stable action profiles.
- $a = \langle a_1, \ldots, a_n \rangle$ is a ("pure strategy") Nash equilibrium iff $\forall i, a_i \in BR(a_{-i})$.

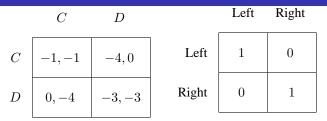
Nash Equilibria of Example Games

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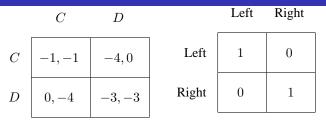
Nash Equilibria of Example Games



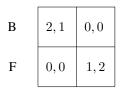
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Nash Equilibria of Example Games



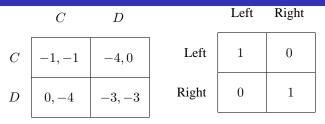
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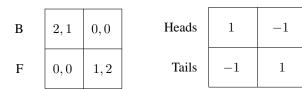
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Nash Equilibria of Example Games



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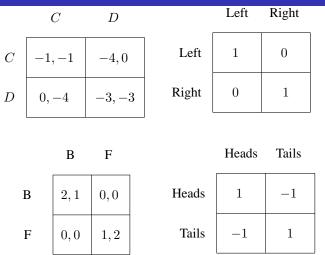
Heads Tails



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Nash Equilibria of Example Games



The paradox of Prisoner's dilemma: the Nash equilibrium is the only non-Pareto-optimal outcome!

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Mixed Strategies

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Mixed Strategies

- It would be a pretty bad idea to play any deterministic strategy in matching pennies
- Idea: confuse the opponent by playing randomly
- Define a strategy s_i for agent i as any probability distribution over the actions A_i .
 - pure strategy: only one action is played with positive probability
 - mixed strategy: more than one action is played with positive probability
 - these actions are called the support of the mixed strategy
- Let the set of all strategies for i be S_i
- Let the set of all strategy profiles be $S = S_1 \times \ldots \times S_n$.

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Utility under Mixed Strategies

- What is your payoff if all the players follow mixed strategy profile $s \in S$?
 - We can't just read this number from the game matrix anymore: we won't always end up in the same cell

Utility under Mixed Strategies

- What is your payoff if all the players follow mixed strategy profile s ∈ S?
 - We can't just read this number from the game matrix anymore: we won't always end up in the same cell
- Instead, use the idea of expected utility from decision theory:

$$u_i(s) = \sum_{a \in A} u_i(a) Pr(a|s)$$
$$Pr(a|s) = \prod_{j \in N} s_j(a_j)$$

Best Response and Nash Equilibrium

Our definitions of best response and Nash equilibrium generalize from actions to strategies.

- Best response:
 - $s_i^* \in BR(s_{-i})$ iff $\forall s_i \in S_i, u_i(s_i^*, s_{-i}) \ge u_i(s_i, s_{-i})$

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- Best response:
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- Nash equilibrium:
 - $s = \langle s_1, \ldots, s_n \rangle$ is a Nash equilibrium iff $\forall i, s_i \in BR(s_{-i})$
- Every finite game has a Nash equilibrium! [Nash, 1950]
 e.g., matching pennies: both players play heads/tails 50%/50%

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Computing Mixed Nash Equilibria: Battle of the Sexes

	В	F
В	2, 1	0,0
F	0,0	1, 2

- It's hard in general to compute Nash equilibria, but it's easy when you can guess the support
- For BoS, let's look for an equilibrium where all actions are part of the support

Computing Mixed Nash Equilibria: Battle of the Sexes



- Let player 2 play B with p, F with 1 p.
- If player 1 best-responds with a mixed strategy, player 2 must make him indifferent between F and B (why?)

Computing Mixed Nash Equilibria: Battle of the Sexes



- Let player 2 play B with p, F with 1 p.
- If player 1 best-responds with a mixed strategy, player 2 must make him indifferent between F and B (why?)

$$u_1(B) = u_1(F)$$

 $2p + 0(1-p) = 0p + 1(1-p)$
 $p = \frac{1}{3}$

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Computing Mixed Nash Equilibria: Battle of the Sexes



- Likewise, player 1 must randomize to make player 2 indifferent.
 - Why is player 1 willing to randomize?

Computing Mixed Nash Equilibria: Battle of the Sexes



- Likewise, player 1 must randomize to make player 2 indifferent.
 - Why is player 1 willing to randomize?
- Let player 1 play B with q, F with 1-q.

$$u_2(B) = u_2(F)$$

$$q + 0(1 - q) = 0q + 2(1 - q)$$

$$q = \frac{2}{3}$$
Thus the mixed strategies $(\frac{2}{3}, \frac{1}{3})$, $(\frac{1}{3}, \frac{2}{3})$ are a Nash equilibrium.

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Interpreting Mixed Strategy Equilibria

What does it mean to play a mixed strategy? Different interpretations:

- Randomize to confuse your opponent
 - consider the matching pennies example
- Players randomize when they are uncertain about the other's action
 - consider battle of the sexes
- Mixed strategies are a concise description of what might happen in repeated play: count of pure strategies in the limit
- Mixed strategies describe population dynamics: 2 agents chosen from a population, all having deterministic strategies. MS is the probability of getting an agent who will play one PS or another.

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