VCG

Lecture 16
Lecture Overview

1. Recap
2. The Groves Mechanism
3. VCG
4. VCG example
5. Individual Rationality
6. Budget Balance
Truthfulness

Definition (Truthfulness)
A quasilinear mechanism is **truthful** if it is direct and $\forall i \forall v_i$, agent $i$’s equilibrium strategy is to adopt the strategy $\hat{v}_i = v_i$.

- Our definition before, adapted for the quasilinear setting
Definition (Efficiency)

A quasilinear mechanism is strictly Pareto efficient, or just efficient, if in equilibrium it selects a choice $x$ such that

$$\forall v \forall x', \sum_i v_i(x) \geq \sum_i v_i(x').$$

- An efficient mechanism selects the choice that maximizes the sum of agents’ utilities, disregarding monetary payments.
- Called economic efficiency to distinguish from other (e.g., computational) notions
- Also called social-welfare maximization
- Note: defined in terms of true (not declared) valuations.
Definition (Budget balance)

A quasilinear mechanism is **budget balanced** when

\[ \forall v, \sum_{i} p_i(s(v)) = 0, \]

where \( s \) is the equilibrium strategy profile.

- regardless of the agents’ types, the mechanism collects and disburses the same amount of money from and to the agents.
- we can also define **weak** or **ex ante** variants.
Definition (Ex interim individual rationality)

A mechanism is ex interim individual rational when
\[ \forall i \forall v_i, \mathbb{E}_{v_{-i}|v_i} v_i(\mathcal{X}(s_i(v_i), s_{-i}(v_{-i}))) - p_i(s_i(v_i), s_{-i}(v_{-i})) \geq 0, \]
where \( s \) is the equilibrium strategy profile.

- no agent loses by participating in the mechanism.
- ex interim because it holds for every possible valuation for agent \( i \), but averages over the possible valuations of the other agents.

Definition (Ex post individual rationality)

A mechanism is ex post individual rational when
\[ \forall i \forall v, v_i(\mathcal{X}(s(v))) - p_i(s(v)) \geq 0, \]
where \( s \) is the equilibrium strategy profile.
Definition (Tractability)

A quasilinear mechanism is \textbf{tractable} when $\forall a \in A$, $\chi(a)$ and $p(a)$ can be computed in polynomial time.

- The mechanism is computationally feasible.
Revenue Maximization

We can also add an objective function to our mechanism. One example: revenue maximization.

**Definition (Revenue maximization)**

A mechanism is **revenue maximizing** when, among the set of functions $\chi$ and $p$ that satisfy the other constraints, the mechanism selects the $\chi$ and $p$ that maximize $\mathbb{E}_\theta \sum_i p_i(s(\theta))$, where $s(\theta)$ denotes the agents’ equilibrium strategy profile.

- The mechanism designer can choose among mechanisms that satisfy the desired constraints by adding an objective function such as revenue maximization.
Revenue Minimization

- The mechanism may not be intended to make money.
- Budget balance may be impossible to satisfy.
- Set weak budget balance as a constraint and add the following objective.

**Definition (Revenue minimization)**

A quasilinear mechanism is revenue minimizing when, among the set of functions $\chi$ and $p$ that satisfy the other constraints, the mechanism selects the $\chi$ and $p$ that minimize $\max_v \sum_i p_i(s(v))$ in equilibrium, where $s(v)$ denotes the agents’ equilibrium strategy profile.

- Note: this considers the worst case over valuations; we could consider average case instead.
Maxmin fairness: make the least-happy agent the happiest.

**Definition (Maxmin fairness)**

A quasilinear mechanism is maxmin fair when, among the set of functions $\chi$ and $p$ that satisfy the other constraints, the mechanism selects the $\chi$ and $p$ that maximize

$$\mathbb{E}_v \left[ \min_{i \in N} v_i(\chi(s(v))) - p_i(s(v)) \right],$$

where $s(v)$ denotes the agents’ equilibrium strategy profile.
When an efficient mechanism is impossible, we may want to get as close as possible.

Minimize the \textbf{worst-case ratio} between optimal social welfare and the social welfare achieved by the given mechanism.

\begin{definition} [Price-of-anarchy minimization]
A quasilinear mechanism minimizes the price of anarchy when, among the set of functions $\chi$ and $p$ that satisfy the other constraints, the mechanism selects the $\chi$ and $p$ that minimize

$$\max_{v \in V} \max_{x \in X} \frac{\sum_{i \in N} v_i(x)}{\sum_{i \in N} v_i(\chi(s(v)))},$$

where $s(v)$ denotes the agents' equilibrium strategy profile in the \textit{worst} equilibrium of the mechanism—i.e., the one in which $\sum_{i \in N} v_i(\chi(s(v)))$ is the smallest.
\end{definition}
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Recall that in the quasilinear utility setting, a mechanism can be defined as a choice rule and a payment rule.

The Groves mechanism is a mechanism that satisfies:
- dominant strategy (truthfulness)
- efficiency

In general it’s not:
- budget balanced
- individual-rational

...though we’ll see later that there’s some hope for recovering these properties.
The Groves Mechanism

Definition (Groves mechanism)

The Groves mechanism is a direct quasilinear mechanism \((\chi, p)\), where

\[
\chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x)
\]

\[
p_i(\hat{v}) = h_i(\hat{v}_{-i}) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))
\]
The Groves Mechanism

\[ \chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x) \]

\[ p_i(\hat{v}) = h_i(\hat{v}_{-i}) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \]

- The choice rule should not come as a surprise (why not?)
The choice rule should not come as a surprise (why not?) because the mechanism is both truthful and efficient: these properties entail the given choice rule.
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So what’s going on with the payment rule?

- the agent $i$ must pay some amount $h_i(\hat{v}_{-i})$ that doesn't depend on his own declared valuation
- the agent $i$ is paid $\sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$, the sum of the others’ valuations for the chosen outcome
Groves Truthfulness

**Theorem**

*Truth telling is a dominant strategy under the Groves mechanism.*

Consider a situation where every agent $j$ other than $i$ follows some arbitrary strategy $\hat{v}_j$. Consider agent $i$’s problem of choosing the best strategy $\hat{v}_i$. As a shorthand, we will write $\hat{v} = (\hat{v}_{-i}, \hat{v}_i)$. The best strategy for $i$ is one that solves

$$\max_{\hat{v}_i} \left( v_i(\chi(\hat{v})) - p(\hat{v}) \right)$$

Substituting in the payment function from the Groves mechanism, we have

$$\max_{\hat{v}_i} \left( v_i(\chi(\hat{v})) - h_i(\hat{v}_{-i}) + \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \right)$$

Since $h_i(\hat{v}_{-i})$ does not depend on $\hat{v}_i$, it is sufficient to solve

$$\max_{\hat{v}_i} \left( v_i(\chi(\hat{v})) + \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \right).$$
Groves Truthfulness

$$\max_{\hat{v}_i} \left( v_i(\chi(\hat{v})) + \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \right).$$

The only way the declaration $\hat{v}_i$ influences this maximization is through the choice of $x$. If possible, $i$ would like to pick a declaration $\hat{v}_i$ that will lead the mechanism to pick an $x \in X$ which solves

$$\max_x \left( v_i(x) + \sum_{j \neq i} \hat{v}_j(x) \right). \quad (1)$$

Under the Groves mechanism,

$$\chi(\hat{v}) = \arg\max_x \left( \sum_i \hat{v}_i(x) \right) = \arg\max_x \left( \hat{v}_i(x) + \sum_{j \neq i} \hat{v}_j(x) \right).$$

The Groves mechanism will choose $x$ in a way that solves the maximization problem in Equation (1) when $i$ declares $\hat{v}_i = v_i$. Because this argument does not depend in any way on the declarations of the other agents, truth-telling is a dominant strategy for agent $i$. 

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Proof intuition

- externalities are internalized
  - agents may be able to change the outcome to another one that they prefer, by changing their declaration
  - however, their utility doesn’t just depend on the outcome—it also depends on their payment
  - since they get paid the (reported) utility of all the other agents under the chosen allocation, they now have an interest in maximizing everyone’s utility rather than just their own
- in general, DS truthful mechanisms have the property that an agent’s payment doesn’t depend on the amount of his declaration, but only on the other agents’ declarations
  - the agent’s declaration is used only to choose the outcome, and to set other agents’ payments
Groves Uniqueness

Theorem (Green–Laffont)

An efficient social choice function \( C : \mathbb{R}^{X^n} \to X \times \mathbb{R}^n \) can be implemented in dominant strategies for agents with unrestricted quasilinear utilities only if \( p_i(v) = h(v_{-i}) - \sum_{j \neq i} v_j(x(v)) \).

- It turns out that the same result also holds for the broader class of Bayes–Nash incentive-compatible efficient mechanisms.
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**Definition (Clarke tax)**

The **Clarke tax** sets the $h_i$ term in a Groves mechanism as

$$h_i(\hat{\nu}_-i) = \sum_{j \neq i} \hat{\nu}_j (\chi(\hat{\nu}_-i)).$$

**Definition (Vickrey-Clarke-Groves (VCG) mechanism)**

The **Vickrey-Clarke-Groves mechanism** is a direct quasilinear mechanism $(\chi, p)$, where

$$\chi(\hat{\nu}) = \arg \max_x \sum_i \hat{\nu}_i(x)$$

$$p_i(\hat{\nu}) = \sum_{j \neq i} \hat{\nu}_j (\chi(\hat{\nu}_-i)) - \sum_{j \neq i} \hat{\nu}_j (\chi(\hat{\nu}))$$
VCG discussion

\[ \chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x) \]

\[ p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}_-i)) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \]

- You get paid everyone’s utility under the allocation that is actually chosen
  - except your own, but you get that directly as utility
- Then you get charged everyone’s utility in the world where you don’t participate
- Thus you pay your social cost
VCG discussion

\[ \chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x) \]

\[ p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}_-i)) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \]

Questions:
- who pays 0?
- agents who don’t affect the outcome
- who pays more than 0? (pivotal) agents who make things worse for others by existing
- who gets paid? (pivotal) agents who make things better for others by existing
VCG discussion

\[ x(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x) \]

\[ p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(x(\hat{v}_{-i})) - \sum_{j \neq i} \hat{v}_j(x(\hat{v})) \]

Questions:
- **who pays 0?**
  - agents who don't affect the outcome
VCG discussion

\[ \chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x) \]

\[ p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}_{-i})) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \]

Questions:
- who pays 0?
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\[ \chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x) \]

\[ p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}-i)) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \]

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Questions:

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- who gets paid?
VCG discussion

$$\chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x)$$

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Questions:

- who pays 0?
  - agents who don’t affect the outcome
- who pays more than 0?
  - (pivotal) agents who make things worse for others by existing
- who gets paid?
  - (pivotal) agents who make things better for others by existing
VCG properties

\[ \chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x) \]

\[ p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v} - i)) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \]

- Because only pivotal agents have to pay, VCG is also called the pivot mechanism.
- It’s dominant-strategy truthful, because it’s a Groves mechanism.
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Selfish routing example

What outcome will be selected by $\chi$?
Selfish routing example

- What outcome will be selected by $\chi$? path $ABEF$. 

![Transportation network graph](image)

- The shortest path taking his declaration into account has a length of 5, and imposes a cost of $-5$ on agents other than him (because it does not involve him). Likewise, the shortest path without AC’s declaration also has a length of 5. Thus, his payment $p_{AC} = (−5) − (−5) = 0$. This is what we expect, since AC is not pivotal.

- Likewise, BD, CE, CF and DF will all pay zero.
Selfish routing example

What outcome will be selected by $x$? path $ABEF$.

How much will $AC$ have to pay?
Selfish routing example

What outcome will be selected by \( \chi \)? path \( AB EF \).

How much will \( AC \) have to pay?

- The shortest path taking his declaration into account has a length of 5, and imposes a cost of \(-5\) on agents other than him (because it does not involve him). Likewise, the shortest path without \( AC \)'s declaration also has a length of 5. Thus, his payment \( p_{AC} = (-5) - (-5) = 0 \).
- This is what we expect, since \( AC \) is not pivotal.
- Likewise, \( BD \), \( CE \), \( CF \) and \( DF \) will all pay zero.
Selfish routing example

How much will $AB$ pay?

![Graph](image-url)
Selfish routing example

How much will $AB$ pay?

- The shortest path taking $AB$’s declaration into account has a length of 5, and imposes a cost of 2 on other agents.
- The shortest path without $AB$ is $ACEF$, which has a cost of 6.
- Thus $p_{AB} = (-6) - (-2) = -4$. 
Selfish routing example

How much will $BE$ pay?

![Transportation network with selfish agents.](image)
Selfish routing example

- How much will $BE$ pay? $p_{BE} = (-6) - (-4) = -2$. 
Selfish routing example

- How much will $BE$ pay? $p_{BE} = (-6) - (-4) = -2$.
- How much will $EF$ pay?
Selfish routing example

- How much will $BE$ pay? $p_{BE} = (-6) - (-4) = -2$.
- How much will $EF$ pay? $p_{EF} = (-7) - (-4) = -3$. 
Selfish routing example

How much will $BE$ pay? $p_{BE} = (-6) - (-4) = -2$.

How much will $EF$ pay? $p_{EF} = (-7) - (-4) = -3$.

$EF$ and $BE$ have the same costs but are paid different amounts. Why?
Selfish routing example

How much will $BE$ pay? $p_{BE} = (-6) - (-4) = -2$.

How much will $EF$ pay? $p_{EF} = (-7) - (-4) = -3$.

- $EF$ and $BE$ have the same costs but are paid different amounts. Why?
- $EF$ has more market power: for the other agents, the situation without $EF$ is worse than the situation without $BE$. 
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Two definitions

Definition (Choice-set monotonicity)
An environment exhibits **choice-set monotonicity** if $\forall i, \ X_{-i} \subseteq X$.

- removing any agent weakly decreases—that is, never increases—the mechanism’s set of possible choices $X$

Definition (No negative externalities)
An environment exhibits **no negative externalities** if $\forall i \forall x \in X_{-i}, \ v_i(x) \geq 0$.

- every agent has zero or positive utility for any choice that can be made without his participation
Example: road referendum

Consider the problem of holding a referendum to decide whether or not to build a road.

- The set of choices is independent of the number of agents, satisfying choice-set monotonicity.
- No agent negatively values the project, though some might value the situation in which the project is not undertaken more highly than the situation in which it is.
Example: simple exchange

Consider a market setting consisting of agents interested in buying a single unit of a good such as a share of stock, and another set of agents interested in selling a single unit of this good. The choices in this environment are sets of buyer-seller pairings (prices are imposed through the payment function).

- If a new agent is introduced into the market, no previously-existing pairings become infeasible, but new ones become possible; thus choice-set monotonicity is satisfied.
- Because agents have zero utility both for choices that involve trades between other agents and no trades at all, there are no negative externalities.
Theorem

The VCG mechanism is ex-post individual rational when the choice set monotonicity and no negative externalities properties hold.

Proof.

All agents truthfully declare their valuations in equilibrium. Then

\[ u_i = v_i(\chi(v)) - \left( \sum_{j \neq i} v_j(\chi(v_i)) - \sum_{j \neq i} v_j(\chi(v)) \right) \]

\[ = \sum_i v_i(\chi(v)) - \sum_{j \neq i} v_j(\chi(v-i)) \]

(2)

\( \chi(v) \) is the outcome that maximizes social welfare, and that this optimization could have picked \( \chi(v-i) \) instead (by choice set monotonicity). Thus,

\[ \sum_{j} v_j(\chi(v)) \geq \sum_{j} v_j(\chi(v-i)). \]
The VCG mechanism is ex-post individual rational when the choice set monotonicity and no negative externalities properties hold.

Proof.

\[ \sum_j v_j(\chi(v)) \geq \sum_j v_j(\chi(v_{-i})) . \]

Furthermore, from no negative externalities,

\[ v_i(\chi(v_{-i})) \geq 0 . \]

Therefore,

\[ \sum_i v_i(\chi(v)) \geq \sum_{j \neq i} v_j(\chi(v_{-i})) , \]

and thus Equation (2) is non-negative.
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Another property

**Definition (No single-agent effect)**

An environment exhibits **no single-agent effect** if \( \forall i, \forall v_{-i}, \forall x \in \arg \max_y \sum_j v_j(y) \) there exists a choice \( x' \) that is feasible without \( i \) and that has \( \sum_{j \neq i} v_j(x') \geq \sum_{j \neq i} v_j(x) \).

**Example**

Consider a single-sided auction. Dropping an agent just reduces the amount of competition, making the others better off.
The VCG mechanism is weakly budget-balanced when the no single-agent effect property holds.

Proof.

Assume truth-telling in equilibrium. We must show that the sum of transfers from agents to the center is greater than or equal to zero.

$$\sum_i p_i(v) = \sum_i \left( \sum_{j \neq i} v_j(\chi(v-i)) - \sum_{j \neq i} v_j(\chi(v)) \right)$$

From the no single-agent effect condition we have that

$$\forall i \sum_{j \neq i} v_j(\chi(v-i)) \geq \sum_{j \neq i} v_j(\chi(v)).$$

Thus the result follows directly.
More good news

Theorem (Krishna & Perry, 1998)

*In any Bayesian game setting in which VCG is ex post individually rational, VCG collects at least as much revenue as any other efficient and ex interim individually-rational mechanism.*

- This is somewhat surprising: does not require dominant strategies, and hence compares VCG to all Bayes–Nash mechanisms.
- A useful corollary: VCG is as budget balanced as any efficient mechanism can be
  - it satisfies weak budget balance in every case where *any* dominant strategy, efficient and *ex interim* IR mechanism would be able to do so.
Bad news

Theorem (Green–Laffont; Hurwicz)

No dominant-strategy incentive-compatible mechanism is always both efficient and weakly budget balanced, even if agents are restricted to the simple exchange setting.

Theorem (Myerson–Satterthwaite)

No Bayes-Nash incentive-compatible mechanism is always simultaneously efficient, weakly budget balanced and ex-interim individual rational, even if agents are restricted to quasilinear utility functions.