Computing Domination; Correlated Equilibria

Lecture 6
Lecture Overview

1. Recap
2. Computational Problems Involving Domination
3. Rationalizability
4. Correlated Equilibrium
5. Computing Correlated Equilibria
Computing equilibria of zero-sum games

\[
\begin{align*}
\text{minimize} & \quad U_1^* \\
\text{subject to} & \quad \sum_{a_2 \in A_2} u_1(a_1, a_2) \cdot s_2^{a_2} \leq U_1^* \quad \forall a_1 \in A_1 \\
& \quad \sum_{a_2 \in A_2} s_2^{a_2} = 1 \\
& \quad s_2^{a_2} \geq 0 \quad \forall a_2 \in A_2
\end{align*}
\]

- This formulation gives us the minmax strategy for player 2.
- To get the minmax strategy for player 1, we need to solve a second (analogous) LP.
To compute a maxmin strategy for player 1 in an arbitrary 2-player game $G$:

- **Create a new game** $G'$ where player 2's payoffs are just the negatives of player 1's payoffs.
- By the minmax theorem, equilibrium strategies for player 1 in $G'$ are equivalent to a maxmin strategies.
- Thus, to find a maxmin strategy for $G$, find an equilibrium strategy for $G'$. 
Let $s_i$ and $s'_i$ be two strategies for player $i$, and let $S_{-i}$ be the set of all possible strategy profiles for the other players.

**Definition**

$s_i$ strictly dominates $s'_i$ if $\forall s_{-i} \in S_{-i}, u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$

**Definition**

$s_i$ weakly dominates $s'_i$ if $\forall s_{-i} \in S_{-i}, u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ and $\exists s_{-i} \in S_{-i}, u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$

**Definition**

$s_i$ very weakly dominates $s'_i$ if $\forall s_{-i} \in S_{-i}, u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$
Iterated Removal of Dominated Strategies

- This process preserves Nash equilibria.
  - strict dominance: all equilibria preserved.
  - weak or very weak dominance: at least one equilibrium preserved.
- Thus, it can be used as a preprocessing step before computing an equilibrium
  - Some games are solvable using this technique.
- What about the order of removal when there are multiple dominated strategies?
  - strict dominance: doesn’t matter.
  - weak or very weak dominance: can affect which equilibria are preserved.
Is $s_i$ strictly dominated by any pure strategy?

Try to identify some pure strategy that is strictly better than $s_i$ for any pure strategy profile of the others.

for all pure strategies $a_i \in A_i$ for player $i$ where $a_i \neq s_i$ do

    dom ← true

    for all pure strategy profiles $a_{-i} \in A_{-i}$ for the players other than $i$ do

        if $u_i(s_i, a_{-i}) \geq u_i(a_i, a_{-i})$ then
            dom ← false
            break
        end if

    end for

    if dom = true then return true

end for

return false
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Computational Problems in Domination

- Identifying strategies **dominated by a pure strategy** (done)
- Identifying strategies **dominated by a mixed strategy**
- Identifying strategies **that survive iterated elimination**
- Asking whether a strategy survives iterated elimination under **all elimination orderings**
- We’ll assume that i’s utility function is strictly positive everywhere (why is this OK?)
Constraints for determining whether $s_i$ is strictly dominated by any mixed strategy

\[ \sum_{j \in A_i} p_j u_i(a_j, a_{-i}) > u_i(s_i, a_{-i}) \quad \forall a_{-i} \in A_{-i} \]

\[ p_j \geq 0 \quad \forall j \in A_i \]

\[ \sum_{j \in A_i} p_j = 1 \]
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• What’s wrong with this program?
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What’s wrong with this program?

- **strict inequality** in the first constraint means we don't have an LP
LP for determining whether $s_i$ is strictly dominated by any mixed strategy

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\text{minimize} & \quad \sum_{j \in A_i} p_j \\
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This is clearly an LP. Why is it a solution to our problem?
LP for determining whether $s_i$ is strictly dominated by any mixed strategy

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- This is clearly an LP. Why is it a solution to our problem?
  - if a solution exists with $\sum_j p_j < 1$ then we can add $1 - \sum_j p_j$ to some $p_k$ and we’ll have a dominating mixed strategy (since utility was assumed to be positive everywhere)
- Our original approach works for very weak domination
- For weak domination we can use that program with a different objective function trick.
Identifying strategies that survive iterated elimination

- This can be done by repeatedly solving our LPs: solving a polynomial number of LPs is still in \( \mathcal{P} \).
  - Checking whether every pure strategy of every player is dominated by any other mixed strategy requires us to solve at worst \( \sum_{i \in N} |A_i| \) linear programs.
  - Each step removes one pure strategy for one player, so there can be at most \( \sum_{i \in N} (|A_i| - 1) \) steps.
  - Thus we need to solve \( O((n \cdot \max_i |A_i|)^2) \) linear programs.
Further questions about iterated elimination

1. (Strategy Elimination) Does there exist some elimination path under which the strategy \( s_i \) is eliminated?

2. (Reduction Identity) Given action subsets \( A'_i \subseteq A_i \) for each player \( i \), does there exist a maximally reduced game where each player \( i \) has the actions \( A'_i \)?

3. (Uniqueness) Does every elimination path lead to the same reduced game?

4. (Reduction Size) Given constants \( k_i \) for each player \( i \), does there exist a maximally reduced game where each player \( i \) has exactly \( k_i \) actions?

For iterated strict dominance these problems are all in \( P \). For iterated weak or very weak dominance these problems are all \( NP \)-complete.
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1. **(Strategy Elimination)** Does there exist some elimination path under which the strategy $s_i$ is eliminated?

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- For **iterated strict dominance** these problems are all in $P$.
- For **iterated weak or very weak dominance** these problems are all $NP$-complete.
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3 Rationalizability

4 Correlated Equilibrium

5 Computing Correlated Equilibria
Rationalizability

- Rather than ask what is irrational, ask what is a best response to some beliefs about the opponent
  - assumes opponent is rational
  - assumes opponent knows that you and the others are rational
  - ...

- Examples
  - is heads rational in matching pennies?
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  - Is heads rational in matching pennies?
  - Is cooperate rational in prisoner’s dilemma?
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Examples
  - is *heads* rational in matching pennies?
  - is *cooperate* rational in prisoner’s dilemma?

Will there always exist a rationalizable strategy?
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- Will there always exist a rationalizable strategy?
  - Yes, equilibrium strategies are always rationalizable.
Rationalizability

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- Examples
  - is heads rational in matching pennies?
  - is cooperate rational in prisoner’s dilemma?

- Will there always exist a rationalizable strategy?
  - Yes, equilibrium strategies are always rationalizable.

- Furthermore, in two-player games, rationalizable $\iff$ survives iterated removal of strictly dominated strategies.
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If there is intelligent life on other planets, in a majority of them, they would have discovered correlated equilibrium before Nash equilibrium.

– Roger Myerson
Examples

- Consider again Battle of the Sexes.
  - Intuitively, the best outcome seems a 50-50 split between $(F, F)$ and $(B, B)$.
  - But there’s no way to achieve this, so either someone loses out (unfair) or both players often miscoordinate.

- Another classic example: traffic game

<table>
<thead>
<tr>
<th></th>
<th>go</th>
<th>wait</th>
</tr>
</thead>
<tbody>
<tr>
<td>go</td>
<td>$-100, -100$</td>
<td>10, 0</td>
</tr>
<tr>
<td>$B$</td>
<td>0, 10</td>
<td>$-10, -10$</td>
</tr>
</tbody>
</table>
Intuition

- What is the natural solution here?
Intuition

- What is the natural solution here?
  - A traffic light: a fair randomizing device that tells one of the agents to go and the other to wait.

- Benefits:
  - the negative payoff outcomes are completely avoided
  - fairness is achieved
  - the sum of social welfare exceeds that of any Nash equilibrium

- We could use the same idea to achieve the fair outcome in battle of the sexes.

- Our example presumed that everyone perfectly observes the random event; not required.

- More generally, some random variable with a commonly known distribution, and a private signal to each player about the outcome.
  - signal doesn’t determine the outcome or others’ signals; however, correlated
Definition (Correlated equilibrium)

Given an $n$-agent game $G = (N, A, u)$, a correlated equilibrium is a tuple $(v, \pi, \sigma)$, where $v$ is a tuple of random variables $v = (v_1, \ldots, v_n)$ with respective domains $D = (D_1, \ldots, D_n)$, $\pi$ is a joint distribution over $v$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a vector of mappings $\sigma_i : D_i \mapsto A_i$, and for each agent $i$ and every mapping $\sigma'_i : D_i \mapsto A_i$ it is the case that

$$\sum_{d \in D} \pi(d) u_i (\sigma_1(d_1), \ldots, \sigma_i(d_i), \ldots, \sigma_n(d_n)) \geq \sum_{d \in D} \pi(d) u_i (\sigma_1(d_1), \ldots, \sigma'_i(d_i), \ldots, \sigma_n(d_n)).$$
Existence

Theorem

For every Nash equilibrium $\sigma^*$ there exists a corresponding correlated equilibrium $\sigma$.

- This is easy to show:
  - let $D_i = A_i$
  - let $\pi(d) = \prod_{i \in N} \sigma^*_i(d_i)$
  - $\sigma_i$ maps each $d_i$ to the corresponding $a_i$.

- Thus, correlated equilibria always exist
Remarks

- Not every correlated equilibrium is equivalent to a Nash equilibrium
  - thus, correlated equilibrium is a weaker notion than Nash
- Any convex combination of the payoffs achievable under correlated equilibria is itself realizable under a correlated equilibrium
  - start with the Nash equilibria (each of which is a CE)
  - introduce a second randomizing device that selects which CE the agents will play
  - regardless of the probabilities, no agent has incentive to deviate
  - the probabilities can be adjusted to achieve any convex combination of the equilibrium payoffs
  - the randomizing devices can be combined
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Computing CE

\[
\sum_{a \in A | a_i = a} p(a)u_i(a) \geq \sum_{a \in A | a_i = a} p(a)u_i(a'_i, a_{-i}) \quad \forall i \in N, \forall a_i, a'_i \in A_i
\]

\[p(a) \geq 0 \quad \forall a \in A\]

\[\sum_{a \in A} p(a) = 1\]

- variables: \(p(a)\); constants: \(u_i(a)\)
Computing CE

\[
\sum_{a \in A | a_i \in a} p(a) u_i(a) \geq \sum_{a \in A | a_i \in a} p(a) u_i(a'_i, a_{-i}) \quad \forall i \in N, \forall a_i, a'_i \in A_i
\]

\[p(a) \geq 0\]

\[\sum_{a \in A} p(a) = 1\]

- variables: \(p(a)\); constants: \(u_i(a)\)
- we could find the social-welfare maximizing CE by adding an objective function

\[
\text{maximize: } \sum_{a \in A} p(a) \sum_{i \in N} u_i(a).
\]
Why are CE easier to compute than NE?

\[ \sum_{a \in A | a_i \in a} p(a)u_i(a) \geq \sum_{a \in A | a_i' \in a} p(a)u_i(a_i', a_{-i}) \quad \forall i \in N, \forall a_i, a_i' \in A_i \]

\[ p(a) \geq 0 \quad \forall a \in A \]

\[ \sum_{a \in A} p(a) = 1 \]

- Intuitively, correlated equilibrium has only a single randomization over outcomes, whereas in NE this is constructed as a product of independent probabilities.

- To change this program so that it finds NE, the first constraint would be

\[ \sum_{a \in A} u_i(a) \prod_{j \in N} p_j(a_j) \geq \sum_{a \in A} u_i(a_i', a_{-i}) \prod_{j \in N \setminus \{i\}} p_j(a_j) \quad \forall i \in N, \forall a_i' \in A_i. \]

- This is a nonlinear constraint!