From Optimality to Equilibrium

Lecture 4
Lecture Overview

1. Recap
2. Pareto Optimality
3. Best Response and Nash Equilibrium
4. Mixed Strategies
Non-Cooperative Game Theory

- **What is it?**
  - mathematical study of interaction between *rational, self-interested* agents

- **Why is it called non-cooperative?**
  - while it’s most interested in situations where agents’ interests conflict, it’s not restricted to these settings
  - the key is that the individual is the basic modeling unit, and that individuals pursue their own interests
    - cooperative/coalitional game theory has teams as the central unit, rather than agents
Defining Games

- Finite, \( n \)-person game: \( \langle N, A, u \rangle \):
  - \( N \) is a finite set of \( n \) players, indexed by \( i \)
  - \( A = A_1 \times \ldots \times A_n \), where \( A_i \) is the action set for player \( i \)
    - \( a \in A \) is an action profile, and so \( A \) is the space of action profiles
  - \( u = \langle u_1, \ldots, u_n \rangle \), a utility function for each player, where \( u_i : A \mapsto \mathbb{R} \)

- Writing a 2-player game as a matrix:
  - row player is player 1, column player is player 2
  - rows are actions \( a \in A_1 \), columns are \( a' \in A_2 \)
  - cells are outcomes, written as a tuple of utility values for each player
Prisoner’s dilemma is any game

\[
\begin{array}{cc}
C & D \\
C & a, a & b, c \\
D & c, b & d, d \\
\end{array}
\]

with \( c > a > d > b \).
Games of Pure Competition

Players have exactly opposed interests

- There must be precisely two players (otherwise they can’t have exactly opposed interests)
- For all action profiles \( a \in A \), \( u_1(a) + u_2(a) = c \) for some constant \( c \)
  - Special case: zero sum

\[
\begin{array}{c|cc}
   & Heads & Tails \\
\hline
Heads & 1     & -1    \\
Tails & -1    & 1     \\
\end{array}
\]
Games of Cooperation

Players have exactly the same interests.

- no conflict: all players want the same things
- \( \forall a \in A, \forall i, j, u_i(a) = u_j(a) \)

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General Games: Battle of the Sexes

The most interesting games combine elements of cooperation and competition.

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1 Recap

2 Pareto Optimality

3 Best Response and Nash Equilibrium

4 Mixed Strategies
We’ve defined some canonical games, and thought about how to play them. Now let’s examine the games from the outside.

From the point of view of an outside observer, can some outcomes of a game be said to be better than others?
Analyzing Games

- We’ve defined some canonical games, and thought about how to play them. Now let’s examine the games from the outside.

- From the point of view of an outside observer, can some outcomes of a game be said to be better than others?
  - We have no way of saying that one agent’s interests are more important than another’s.
  - Intuition: imagine trying to find the revenue-maximizing outcome when you don’t know what currency has been used to express each agent’s payoff.

- Are there situations where we can still prefer one outcome to another?
Pareto Optimality

**Idea:** sometimes, one outcome $o$ is at least as good for every agent as another outcome $o'$, and there is some agent who strictly prefers $o$ to $o'$

- in this case, it seems reasonable to say that $o$ is better than $o'$
- we say that $o$ Pareto-dominates $o'$.
Pareto Optimality

- **Idea**: sometimes, one outcome $o$ is at least as good for every agent as another outcome $o'$, and there is some agent who strictly prefers $o$ to $o'$
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- An outcome $o^*$ is **Pareto-optimal** if there is no other outcome that Pareto-dominates it.
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An outcome $o^*$ is **Pareto-optimal** if there is no other outcome that Pareto-dominates it.

- can a game have more than one Pareto-optimal outcome?
Pareto Optimality

**Idea:** sometimes, one outcome $o$ is at least as good for every agent as another outcome $o'$, and there is some agent who strictly prefers $o$ to $o'$

- in this case, it seems reasonable to say that $o$ is better than $o'$
- we say that $o$ *Pareto-dominates* $o'$.

An outcome $o^*$ is **Pareto-optimal** if there is no other outcome that Pareto-dominates it.

- can a game have more than one Pareto-optimal outcome?
- does every game have at least one Pareto-optimal outcome?
### Pareto Optimal Outcomes in Example Games

Consider the following game matrix:

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This game matrix represents a scenario where players have two strategies: **C** (Cooperate) and **D** (Defect). The outcomes are given as pairs representing the payoffs for each player. For example, if both players cooperate, each gets -1; if one cooperates and the other defects, the defector gets 0 and the cooperator gets -4. The payoffs are negative because they represent delays or costs incurred by each player.

**Note:** This is a simplified example to illustrate the concept of Pareto optimality in game theory. In real-world scenarios, the payoffs and strategies can be much more complex.
## Pareto Optimal Outcomes in Example Games

The following table and diagram illustrate the concept of Pareto optimality in example games:

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### Diagram:

- **Left**
  - 1
  - 0

- **Right**
  - 0
  - 1

This example game demonstrates how to identify Pareto optimal outcomes.
Pareto Optimal Outcomes in Example Games

\[\begin{array}{cc|cc}
 & C & D \\
\hline
C & -1, -1 & -4, 0 \\
D & 0, -4 & -3, -3 \\
\end{array}\]

\[\begin{array}{cc|cc}
 & Left & Right \\
\hline
Left & 1 & 0 \\
Right & 0 & 1 \\
\end{array}\]

\[\begin{array}{cc|cc}
 & B & F \\
\hline
B & 2, 1 & 0, 0 \\
F & 0, 0 & 1, 2 \\
\end{array}\]
### Pareto Optimal Outcomes in Example Games

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Best Response

- If you knew what everyone else was going to do, it would be easy to pick your own action
Best Response

- If you knew what everyone else was going to do, it would be easy to pick your own action
- Let $a_{-i} = \langle a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \rangle$.
  - now $a = (a_{-i}, a_i)$

Best response: $a_i^* \in BR(a_{-i})$ iff
$$\forall a_i \in A_i, \ u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i})$$
Nash Equilibrium

- Now let’s return to the setting where no agent knows anything about what the others will do.
- What can we say about which actions will occur?
Now let’s return to the setting where no agent knows anything about what the others will do.

What can we say about which actions will occur?

Idea: look for stable action profiles.

\( a = \langle a_1, \ldots, a_n \rangle \) is a (‘‘pure strategy’’) Nash equilibrium iff

\[ \forall i, a_i \in BR(a_{-i}). \]
### Nash Equilibria of Example Games

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*The paradox of [Prisoner's dilemma]: the Nash equilibrium is the only non-Pareto-optimal outcome!*
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Mixed Strategies

- It would be a pretty bad idea to play any deterministic strategy in matching pennies.
- Idea: confuse the opponent by playing randomly.
- Define a strategy $s_i$ for agent $i$ as any probability distribution over the actions $A_i$.
  - **pure strategy**: only one action is played with positive probability.
  - **mixed strategy**: more than one action is played with positive probability.
    - these actions are called the support of the mixed strategy.
- Let the set of all strategies for $i$ be $S_i$.
- Let the set of all strategy profiles be $S = S_1 \times \ldots \times S_n$. 
Utility under Mixed Strategies

- What is your **payoff** if all the players follow mixed strategy profile \( s \in S \)?
- We can’t just read this number from the game matrix anymore: we won’t always end up in the same cell
Utility under Mixed Strategies

- What is your **payoff** if all the players follow mixed strategy profile \( s \in S \)?
  - We can’t just read this number from the game matrix anymore: we won’t always end up in the same cell.
- Instead, use the idea of **expected utility** from decision theory:

\[
    u_i(s) = \sum_{a \in A} u_i(a) Pr(a \mid s)
\]

\[
    Pr(a \mid s) = \prod_{j \in N} s_j(a_j)
\]
Best Response and Nash Equilibrium

Our definitions of best response and Nash equilibrium generalize from actions to strategies.

- **Best response:**
  \[ s_i^* \in BR(s_{-i}) \text{ iff } \forall s_i \in S_i, \ u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \]

- Nash equilibrium: \[ s = \langle s_1, \ldots, s_n \rangle \text{ is a Nash equilibrium iff } \forall i, s_i \in BR(s_{-i}) \]

Every finite game has a Nash equilibrium! [Nash, 1950]

E.g., matching pennies: both players play heads/tails 50%/50%
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- **Every finite game has a Nash equilibrium!** [Nash, 1950]
  - e.g., matching pennies: both players play heads/tails 50%/50%
Computing Mixed Nash Equilibria: Battle of the Sexes

- It’s hard in general to compute Nash equilibria, but it’s easy when you can guess the support.
- For BoS, let’s look for an equilibrium where all actions are part of the support.
Computing Mixed Nash Equilibria: Battle of the Sexes

Let player 2 play $B$ with $p$, $F$ with $1 - p$.

If player 1 best-responds with a mixed strategy, player 2 must make him indifferent between $F$ and $B$ (why?)
Computing Mixed Nash Equilibria: Battle of the Sexes

- Let player 2 play \( B \) with \( p \), \( F \) with \( 1 - p \).
- If player 1 best-responds with a mixed strategy, player 2 must make him indifferent between \( F \) and \( B \) (why?)

\[
\begin{align*}
    u_1(B) &= u_1(F) \\
    2p + 0(1 - p) &= 0p + 1(1 - p) \\
    2p &= 1 - p \\
    p &= \frac{1}{3}
\end{align*}
\]
Computing Mixed Nash Equilibria: Battle of the Sexes

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Likewise, player 1 must randomize to make player 2 indifferent.

- Why is player 1 willing to randomize?
Computing Mixed Nash Equilibria: Battle of the Sexes

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- Likewise, player 1 must randomize to make player 2 indifferent.
- Why is player 1 willing to randomize?
- Let player 1 play $B$ with $q$, $F$ with $1 - q$.

$$u_2(B) = u_2(F)$$

$$q + 0(1 - q) = 0q + 2(1 - q)$$

$$q = \frac{2}{3}$$

- Thus the mixed strategies $(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})$ are a Nash equilibrium.
Interpreting Mixed Strategy Equilibria

What does it mean to play a mixed strategy? Different interpretations:

- Randomize to **confuse** your opponent
  - consider the matching pennies example
- Players randomize when they are **uncertain** about the other’s action
  - consider battle of the sexes
- Mixed strategies are a concise description of what might happen in **repeated play**: count of pure strategies in the limit
- Mixed strategies describe **population dynamics**: 2 agents chosen from a population, all having deterministic strategies. MS is the probability of getting an agent who will play one PS or another.