Multiunit Auctions

Lecture 21

Lecture Overview

1. Recap
2. Simple Multiunit Auctions
3. Unlimited Supply
4. General Multiunit Auctions

Designing optimal auctions

Definition (virtual valuation)

Bidder i's virtual valuation is $\psi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$.

Definition (bidder-specific reserve price)

Bidder i's bidder-specific reserve price r_i^* is the value for which $\psi_i(r_i^*) = 0$.

Theorem

The optimal (single-good) auction is a sealed-bid auction in which every agent is asked to declare his valuation. The good is sold to the agent $i = \arg\max_i \psi_i(\hat{v}_i)$, as long as $v_i > r_i^*$. If the good is sold, the winning agent i is charged the smallest valuation that he could have declared while still remaining the winner: $\inf\{v_i^* : \psi_i(v_i^*) \geq 0$ *and* $\forall j \neq i,\ \psi_i(v_i^*) \geq \psi_j(\hat{v}_j)\}$.

Analyzing optimal auctions

Optimal Auction:

- winning agent: $i = \arg\max_i \psi_i(\hat{v}_i)$, as long as $v_i > r_i^*$.
- i is charged the smallest valuation that he could have declared while still remaining the winner, $\inf\{v_i^* : \psi_i(v_i^*) \geq 0$ and $\forall j \neq i,\ \psi_i(v_i^*) \geq \psi_j(\hat{v}_j)\}$.

- it's a second-price auction with a reserve price, held in virtual valuation space.
- neither the reserve prices nor the virtual valuation transformation depends on the agent's declaration
- thus the proof that a second-price auction is dominant-strategy truthful applies here as well.

Going beyond IPV

- common value model
  - motivation: oil well
  - winner's curse
  - things can be improved by revealing more information
- general model
  - IPV + common value
  - example motivation: private value plus resale

Risk Attitudes

What kind of auction would the auctioneer prefer?

- Buyer is not risk neutral:
  - no change under various risk attitudes for second price
  - in first-price, increasing bid amount increases probability of winning, decreases profit. This is good for risk-averse bidder, bad for risk-seeking bidder.
  - Risk averse, IPV: First $\succ$ [Japanese $=$ English $=$ Second]
  - Risk seeking, IPV: Second $\succ$ First
- Auctioneer is not risk neutral:
  - revenue is fixed in first-price auction (the expected amount of the second-highest bid)
  - revenue varies in second-price auction, with the same expected value
  - thus, a risk-averse seller prefers first-price to second-price.

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  - every unit is sold for the amount of the $k+1$st highest bid
- how else can we sell the goods?
  - pay-your-bid: "discriminatory" pricing, because bidders will pay different amounts for the same thing
  - lowest winning bid: very similar to VCG, but ensures that bidders don't pay zero if there are fewer bids than units for sale
  - sequential single-good auctions

Revenue Equivalence

Theorem (Revenue equivalence theorem, multiunit version)

Assume that each of n risk-neutral agents has an independent private valuation for a single unit of k identical goods at auction, drawn from a common cumulative distribution $F(v)$ that is strictly increasing and atomless on $[\underline{v}, \bar{v}]$. Then any efficient auction mechanism in which any agent with valuation $\underline{v}$ has an expected utility of zero yields the same expected revenue, and hence results in any bidder with valuation v_i making the same expected payment.

Sequential Auctions

Although we can apply the revelation principle, for greater intuition we can also use backward induction to derive the equilibrium strategies in finitely-repeated second-price auctions.

- everyone should bid honestly in the final auction
- we can also compute a bidder's expected utility (conditioned on type) in that auction
- in the second-last auction, bid the difference between valuation and the expected utility for losing
  - i.e., bid valuation minus the expected utility for playing the second auction
  - why: consider affine transformation of valuations subtracting this constant expected utility
- combining these last two auctions together, there's some expected utility to playing both of them
- now this is the "expected utility of losing"
- apply backward induction

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Unlimited Supply

- consider MP3 downloads as an example of a multiunit good.
- They differ from the other examples we gave:
  - the seller can produce additional units at zero marginal cost
  - hence has an effectively unlimited supply of the good
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- How should such goods be auctioned?
  - the seller will have to artificially reduce supply
  - the *first* unit of the good might have been very expensive to produce!

Optimal Single Price

If we *knew* bidders' valuations but had to offer the goods at the same price to all bidders, it would be easy to compute the optimal single price.

Definition (Optimal single price)

The optimal single price is calculated as follows.

1. Order the bidders in descending order of valuation; let v_i denote the ith-highest valuation.
2. Calculate $opt \in \arg\max_{i \in \{1,\ldots,n\}} i \cdot v_i$.
3. The optimal single price is v_{opt}.

Random Sampling Auction

Definition (Random sampling optimal price auction)

The *random sampling optimal price auction* is defined as follows.

1. Randomly partition the set of bidders N into two sets, N_1 and N_2 (i.e., $N = N_1 \cup N_2$; $N_1 \cap N_2 = \emptyset$; each bidder has probability 0.5 of being assigned to each set).
2. Using the procedure above find p_1 and p_2, where p_i is the optimal single price to charge the set of bidders N_i.
3. Then set the allocation and payment rules as follows:
   - For each bidder $i \in N_1$, award a unit of the good if and only if $b_i \geq p_2$, and charge the bidder p_2;
   - For each bidder $j \in N_2$, award a unit of the good if and only if $b_j \geq p_1$, and charge the bidder p_1.

Results

Theorem

Random sampling optimal price auctions are dominant-strategy truthful, weakly budget balanced and ex post individually rational.

Theorem

The random sampling optimal price auction always yields expected revenue that is at least a $(\frac{1}{4.68})$ *constant fraction of the revenue that would be achieved by charging bidders the optimal single price, subject to the constraint that at least two units of the good must be sold.*

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Multiunit Demand

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How does VCG behave when (some) bidders may want more than a single unit of the good?

- no longer a $k+1$st-price auction
- instead, all winning bidders who won the same number of units will pay the same amount as each other.
  - the change in social welfare from dropping any of these bidders is the same.
- Bidders who win different numbers of units will not necessarily pay the same per unit prices.
- However, bidders who win larger numbers of units will pay at least as much in total (not necessarily per unit) as bidders who won smaller numbers of units
  - their impact on social welfare will always be at least as great

Winner Determination for Multiunit Demand

- Let m be the number of units available, and let $\hat{v}_i(k)$ denote bidder i's declared valuation for being awarded k units.
- It's no longer computationally easy to identify the winners—now it's a (NP-complete) weighted knapsack problem:

$$\begin{aligned}
\text{maximize} \quad & \sum_{i \in N} \sum_{1 \le k \le m} \hat{v}_i(k) x_{k,i} && (1)\\
\text{subject to} \quad & \sum_{i \in N} \sum_{1 \le k \le m} k \cdot x_{k,i} \le m && (2)\\
& \sum_{1 \le k \le m} x_{k,i} \le 1 \qquad \forall i \in N && (3)\\
& x_{k,i} = \{0,1\} \qquad \forall 1 \le k \le m, i \in N && (4)
\end{aligned}$$

Winner Determination for Multiunit Demand

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$$x_{k,i} = \{0, 1\} \qquad \forall 1 \leq k \leq m, i \in N \tag{4}$$

- $x_{k,i}$ indicates whether bidder i is allocated exactly k units
- maximize: sum of agents' valuations for the chosen allocation
- (2): number of units allocated does not exceed number available
- (3): no more than one $x_{\cdot,i}$ is nonzero for any i
- (4): all x's must be integers

Multiunit Valuations

How can bidders express their valuations in a multiunit auction?

- m homogeneous goods, let S denote some set
- general: let $p_1, \ldots, p_m$ be arbitrary, non-negative real numbers. Then $v(S) = \sum_{j=1}^{|S|} p_j$.
- downward sloping: general, but $p_1 \geq p_2 \geq \ldots \geq p_m$
- additive: $v(S) = c|S|$
- single-item: $v(S) = c$ if $s \neq \emptyset$; 0 otherwise
- fixed-budget: $v(S) = \min(c|S|, b)$
- majority: $v(S) = c$ if $|S| \geq m/2$, 0 otherwise