Lecture Overview

1 Recap
2 General Multiunit Auctions
3 Combinatorial Auctions
4 Bidding Languages
Designing optimal auctions

**Definition (virtual valuation)**

Bidder $i$’s virtual valuation is $\psi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$.

**Definition (bidder-specific reserve price)**

Bidder $i$’s bidder-specific reserve price $r^*_i$ is the value for which $\psi_i(r^*_i) = 0$.

**Theorem**

The optimal (single-good) auction is a sealed-bid auction in which every agent is asked to declare his valuation. The good is sold to the agent $i = \arg \max_i \psi_i(\hat{v}_i)$, as long as $v_i > r^*_i$. If the good is sold, the winning agent $i$ is charged the smallest valuation that he could have declared while still remaining the winner: $\inf \{v_i^*: \psi_i(v_i^*) \geq 0 \text{ and } \forall j \neq i, \psi_i(v_i^*) \geq \psi_j(\hat{v}_j)\}$. 
Analyzing optimal auctions

Optimal Auction:

- winning agent: \( i = \arg \max_i \psi_i(\hat{v}_i) \), as long as \( v_i > r_i^* \).
- \( i \) is charged the smallest valuation that he could have declared while still remaining the winner,
  \( \inf \{ v_i^* : \psi_i(v_i^*) \geq 0 \text{ and } \forall j \neq i, \psi_i(v_i^*) \geq \psi_j(\hat{v}_j) \} \).

- it’s a second-price auction with a reserve price, held in virtual valuation space.
- neither the reserve prices nor the virtual valuation transformation depends on the agent’s declaration
- thus the proof that a second-price auction is dominant-strategy truthful applies here as well.
Going beyond IPV

- common value model
  - motivation: oil well
  - winner’s curse
  - things can be improved by revealing more information
- general model
  - IPV + common value
  - example motivation: private value plus resale
Risk Attitudes

What kind of auction would the auctioneer prefer?

- **Buyer is not risk neutral:**
  - no change under various risk attitudes for second price
  - in first-price, increasing bid amount increases probability of winning, decreases profit. This is good for risk-averse bidder, bad for risk-seeking bidder.
  - Risk averse, IPV: First ≻ [Japanese = English = Second]
  - Risk seeking, IPV: Second ≻ First

- **Auctioneer is not risk neutral:**
  - revenue is fixed in first-price auction (the expected amount of the second-highest bid)
  - revenue varies in second-price auction, with the same expected value
  - thus, a risk-averse seller prefers first-price to second-price.
Multiunit Auctions

- now let’s consider a setting in which
  - there are $k$ identical goods for sale in a single auction
  - every bidder only wants one unit
- **VCG** in this setting:
  - every unit is sold for the amount of the $k + 1$st highest bid
- revenue equivalence holds here, so all other methods of setting prices lead to the same payments in equilibrium.
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Multiunit Demand

How does VCG behave when (some) bidders may want more than a single unit of the good?
Multiunit Demand

How does VCG behave when (some) bidders may want more than a single unit of the good?

- no longer a $k + 1$st-price auction
- instead, all winning bidders who won the same number of units will pay the same amount as each other.
  - the change in social welfare from dropping any of these bidders is the same.
- Bidders who win different numbers of units will not necessarily pay the same per unit prices.
- However, bidders who win larger numbers of units will pay at least as much in total (not necessarily per unit) as bidders who won smaller numbers of units.
  - their impact on social welfare will always be at least as great.
Winner Determination for Multiunit Demand

- Let $m$ be the number of units available, and let $\hat{v}_i(k)$ denote bidder $i$’s declared valuation for being awarded $k$ units.

- It’s no longer computationally easy to identify the winners—now it’s a (NP-complete) weighted knapsack problem:

\[
\text{maximize} \quad \sum_{i \in N} \sum_{1 \leq k \leq m} \hat{v}_i(k) x_{k,i} \quad \text{(1)}
\]

subject to

\[
\sum_{i \in N} \sum_{1 \leq k \leq m} k \cdot x_{k,i} \leq m \quad \text{(2)}
\]

\[
\sum_{1 \leq k \leq m} x_{k,i} \leq 1 \quad \forall i \in N \quad \text{(3)}
\]

\[
x_{k,i} = \{0, 1\} \quad \forall 1 \leq k \leq m, i \in N \quad \text{(4)}
\]
Winner Determination for Multiunit Demand

maximize \[ \sum_{i \in N} \sum_{1 \leq k \leq m} \hat{v}_i(k) x_{k,i} \] 
subject to \[ \sum_{i \in N} \sum_{1 \leq k \leq m} k \cdot x_{k,i} \leq m \] 
\[ \sum_{1 \leq k \leq m} x_{k,i} \leq 1 \quad \forall i \in N \] 
\[ x_{k,i} = \{0, 1\} \quad \forall 1 \leq k \leq m, i \in N \]

- \( x_{k,i} \) indicates whether bidder \( i \) is allocated exactly \( k \) units
- maximize: sum of agents' valuations for the chosen allocation
- (2): number of units allocated does not exceed number available
- (3): no more than one \( x_{.,i} \) is nonzero for any \( i \)
- (4): all \( x \)'s must be integers
Multiunit Valuations

How can bidders express their valuations in a multiunit auction?

- **$m$** homogeneous goods, let $S$ denote some set
- **general**: let $p_1, \ldots, p_m$ be arbitrary, non-negative real numbers. Then $v(S) = \sum_{j=1}^{|S|} p_j$.
- **downward sloping**: general, but $p_1 \geq p_2 \geq \ldots \geq p_m$
- **additive**: $v(S) = c|S|$
- **single-item**: $v(S) = c$ if $s \neq \emptyset$; 0 otherwise
- **fixed-budget**: $v(S) = \min(c|S|, b)$
- **majority**: $v(S) = c$ if $|S| \geq m/2$, 0 otherwise
Advanced Multiunit Auctions

- **Unlimited supply**: random sampling auctions
  - how to sell goods that cost nothing to produce, when the valuation distribution is unknown?

- **Search engine advertising**: position auctions
  - how to sell slots on the right-hand side of internet search results
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now consider a case where multiple, heterogeneous goods are being sold.

consider the sorts of valuations that agents could have in this case:

- **complementarity**: for sets $S$ and $T$, $v(S \cup T) > v(S) + v(T)$
  - e.g., a left shoe and a right shoe
- **substitutability**: $v(S \cup T) < v(S) + v(T)$
  - e.g., two tickets to different movies playing at the same time

substitutability is relatively easy to deal with
  - e.g., just sell the goods sequentially, or allow bid withdrawal

complementarity is trickier...
Fun Game

- 9 plots of land for sale, geographically related as shown
- IPV, normally distributed with mean 50, stdev 5
- Payoff:
  - if you get one good other than #5: $v_i$
  - any two goods: $3v_i$
  - any three (or more) goods: $5v_i$
- Rules:
  - auctioneer moves from one good to the next sequentially, holding an English auction for each good.
  - bidding stops on a good: move on to the next good
  - no bids for any of the 9 goods: end the auction
Combinatorial auctions

- running a simultaneous ascending auction is inefficient
  - exposure problem
  - inefficiency due to fear of exposure
- if we want an efficient outcome, why not just run VCG?
  - unfortunately, it again requires solving an NP-complete problem
- let there be $n$ goods, $m$ bids, sets $C_j$ of XOR bids
- weighted set packing problem:

$$
\max \sum_{i=1}^{m} x_i p_i \\
\text{subject to } \sum_{i|g \in S_i} x_i \leq 1 \quad \forall g \\
x_i \in \{0, 1\} \quad \forall i \\
\sum_{k \in C_j} x_k \leq 1 \quad \forall j
$$
Combinatorial auctions

\[
\begin{align*}
\text{max} & \quad \sum_{i=1}^{m} x_i p_i \\
\text{subject to} & \quad \sum_{i \mid g \in S_i} x_i \leq 1 \quad \forall g \\
& \quad x_i \in \{0, 1\} \quad \forall i \\
& \quad \sum_{k \in C_j} x_k \leq 1 \quad \forall j
\end{align*}
\]

- we don’t need the XOR constraints
- instead, we can introduce “dummy goods” that don’t correspond to goods in the auction, but that enforce XOR constraints.
- amounts to exactly the same thing: the first constraint has the same form as the third
Winner determination problem

How do we deal with the computational complexity of the winner determination problem?

- Require bids to come from a restricted set, guaranteeing that the WDP can be solved in polynomial time
  - problem: these restricted sets are very restricted...
- Use heuristic methods to solve the problem
  - this works pretty well in practice, making it possible to solve WDPs with many hundreds of goods and thousands of bids.
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Expressing a bid in combinatorial auctions: OR bidding

- **Atomic bid:** $(S, p)$ means $v(S) = p$
- implicitly, an “AND” of the singletons in $S$

- **OR bid:** combine atomic bids

- let $v_1, v_2$ be arbitrary valuations

\[ (v_1 \lor v_2)(S) = \max_{R, T \subseteq S} \left[ v_1(R) + v_2(S) \right] \quad \text{subject to} \quad R \cup T = \emptyset \]

**Theorem**

*OR bids can express all valuations that do not have any substitutability, and only these valuations.*
XOR Bids

- **XOR bidding**: allow substitutabilities
  - \((v_1 \text{XOR} v_2)(S) = \max(v_1(S), v_2(S))\)

**Theorem**

*XOR bids can represent any valuation*

- this isn’t really surprising, since we can enumerate valuations
- however, this implies that they don’t represent everything efficiently

**Theorem**

*Additive valuations require linear space with OR, exponential space with XOR*

- likewise with many other valuations: any in which the price is different for every bundle
Composite Bidding Languages

- OR-of-XOR
  - sets of XOR bids, where the bidder is willing to get either one or zero from each set
    - \((\ldots \text{XOR} \ldots \text{XOR} \ldots) \text{OR} (\ldots) \text{OR} (\ldots)\)

**Theorem**

*Any downward sloping valuation can be represented using the OR-of-XOR language using at most \(m^2\) atomic bids.*

- XOR-of-OR
  - a set of OR atomic bids, where the bidder is willing to select from only one of these sets
- generalized OR/XOR
  - arbitrary nesting of OR and XOR
The OR* Language

- OR*
  - OR, but uses dummy goods to simulate XOR constraints

**Theorem**

\[ \text{OR-of-XOR size } k \implies \text{OR* size } k, \leq k \text{ dummy goods} \]

**Theorem**

\[ \text{Generalized OR/XOR size } k \implies \text{OR* size } k, \leq k^2 \text{ dummy goods} \]

**Corollary**

\[ \text{XOR-of-OR size } k \implies \text{OR* size } k, \leq k^2 \text{ dummy goods} \]
Advanced topics in combinatorial auctions

• **iterative combinatorial auction mechanisms**
  - reduce the amount bidders have to disclose / communication complexity
  - allow bidders to learn about each others’ valuations: e.g., affiliated values

• **non-VCG mechanisms** for restricted valuation classes
  - these can rely on polynomial-time winner determination algorithms