Quasilinear Mechanisms; Groves Mechanism

Lecture 15
Lecture Overview

1. Recap
2. Quasilinear Mechanisms
3. Properties
4. The Groves Mechanism
It turns out that truthfulness can always be achieved!
Consider an arbitrary, non-truthful mechanism (e.g., may be indirect)
Recall that a mechanism defines a game, and consider an equilibrium \( s = (s_1, \ldots, s_n) \)
We can construct a new direct mechanism, as shown above.

This mechanism is truthful by exactly the same argument that $s$ was an equilibrium in the original mechanism.

“The agents don’t have to lie, because the mechanism already lies for them.”
Impossibility Result

Theorem (Gibbard-Satterthwaite)

Consider any social choice function $C$ of $N$ and $O$. If:

1. $|O| \geq 3$ (there are at least three outcomes);
2. $C$ is onto; that is, for every $o \in O$ there is a preference profile $\succ$ such that $C(\succ) = o$ (this property is sometimes also called citizen sovereignty); and
3. $C$ is dominant-strategy truthful,

then $C$ is dictatorial.
Definition (Quasilinear preferences)

Agents have quasilinear preferences in an $n$-player Bayesian game when the set of outcomes is

$$O = X \times \mathbb{R}^n$$

for a finite set $X$, and the utility of an agent $i$ given joint type $\theta$ is given by

$$u_i(o, \theta) = u_i(x, \theta) - f_i(p_i),$$

where $o = (x, p)$ is an element of $O$, $u_i : X \times \Theta \mapsto \mathbb{R}$ is an arbitrary function and $f_i : \mathbb{R} \mapsto \mathbb{R}$ is a strictly monotonically increasing function.
Risk Neutrality

(a) Risk neutrality

(b) Risk neutrality: fair lottery

Figure 8.3
Risk attitudes: Risk aversion, risk neutrality, risk seeking, and in each case, utility for the outcomes of a fair lottery.

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Risk Aversion

(c) Risk aversion

(d) Risk aversion: fair lottery
Risk Seeking

(e) Risk seeking

(f) Risk seeking: fair lottery
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**Definition (Quasilinear mechanism)**

A mechanism in the quasilinear setting (for a Bayesian game setting \((N, O = X \times \mathbb{R}^n, \Theta, p, u)\)) is a triple \((A, \chi, p)\), where

- \(A = A_1 \times \cdots \times A_n\), where \(A_i\) is the set of actions available to agent \(i \in N\),
- \(\chi : A \mapsto \Pi(X)\) maps each action profile to a distribution over choices, and
- \(p : A \mapsto \mathbb{R}^n\) maps each action profile to a payment for each agent.
Direct Quasilinear Mechanism

Definition (Direct quasilinear mechanism)

A **direct quasilinear mechanism** (for a Bayesian game setting \((N, O = X \times \mathbb{R}^n, \Theta, p, u)\)) is a pair \((\chi, p)\). It defines a standard mechanism in the quasilinear setting, where for each \(i\), \(A_i = \Theta_i\).

Definition (Conditional utility independence)

A Bayesian game exhibits **conditional utility independence** if for all agents \(i \in N\), for all outcomes \(o \in O\) and for all pairs of joint types \(\theta\) and \(\theta' \in \Theta\) for which \(\theta_i = \theta'_i\), it holds that \(u_i(o, \theta) = u_i(o, \theta')\).
Quasilinear Mechanisms with Conditional Utility Independence

- Given conditional utility independence, we can write $i$’s utility function as $u_i(o, \theta_i)$
  - it does not depend on the other agents’ types
- An agent’s **valuation** for choice $x \in X$: $v_i(x) = u_i(x, \theta_i)$
  - the maximum amount $i$ would be willing to pay to get $x$
  - in fact, $i$ would be indifferent between keeping the money and getting $x$
- Alternate definition of **direct mechanism**:
  - ask agents $i$ to declare $v_i(x)$ for each $x \in X$
- Define $\hat{v}_i$ as the valuation that agent $i$ declares to such a direct mechanism
  - may be different from his true valuation $v_i$
- Also define the tuples $\hat{v}$, $\hat{v}_{-i}$
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Definition (Truthfulness)

A quasilinear mechanism is truthful if it is direct and \( \forall i \forall v_i \), agent \( i \)'s equilibrium strategy is to adopt the strategy \( \hat{v}_i = v_i \).

- Our definition before, adapted for the quasilinear setting
A quasilinear mechanism is strictly Pareto efficient, or just efficient, if in equilibrium it selects a choice $x$ such that

$$\forall v \forall x', \sum_i v_i(x) \geq \sum_i v_i(x').$$

- An efficient mechanism selects the choice that maximizes the sum of agents’ utilities, disregarding monetary payments.
- How is this related to Pareto efficiency from GT?
A quasilinear mechanism is strictly Pareto efficient, or just efficient, if in equilibrium it selects a choice \( x \) such that

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- An efficient mechanism selects the choice that maximizes the sum of agents’ utilities, disregarding monetary payments.
- How is this related to Pareto efficiency from GT?
  - if we include the mechanism as an agent, all Pareto-efficient outcomes involve the same choice (and different payments)
  - any outcome involving another choice is Pareto-dominated: some agents could make a side-payment to others such that all would prefer the swap
A quasilinear mechanism is strictly Pareto efficient, or just efficient, if in equilibrium it selects a choice \( x \) such that

\[
\forall v \forall x', \sum_i v_i(x) \geq \sum_i v_i(x').
\]

Called economic efficiency to distinguish from other (e.g., computational) notions

Also called social-welfare maximization

Note: defined in terms of true (not declared) valuations.
A quasilinear mechanism is **budget balanced** when

\[ \forall v, \sum_i p_i(s(v)) = 0, \]

where \( s \) is the equilibrium strategy profile.

- regardless of the agents’ types, the mechanism collects and disburses the same amount of money from and to the agents
Definition (Budget balance)

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- regardless of the agents’ types, the mechanism collects and disburses the same amount of money from and to the agents
- relaxed version: **weak budget balance**:

\[ \forall v, \sum_i p_i(s(v)) \geq 0 \]

- the mechanism never takes a loss, but it may make a profit
Budget Balance

Definition (Budget balance)

A quasilinear mechanism is budget balanced when

$$\forall v, \sum_i p_i(s(v)) = 0,$$

where $s$ is the equilibrium strategy profile.

- regardless of the agents’ types, the mechanism collects and disburses the same amount of money from and to the agents
- Budget balance can be required to hold \textit{ex ante}:

$$E_v \sum_i p_i(s(v)) = 0$$

- the mechanism must break even or make a profit only on expectation
Individual-Rationality

Definition (*Ex interim* individual rationality)

A mechanism is *ex interim* individual rational when
\[ \forall i \forall v_i, \mathbb{E}_{v_{-i}|v_i} v_i(\chi(s_i(v_i), s_{-i}(v_{-i}))) - p_i(s_i(v_i), s_{-i}(v_{-i})) \geq 0, \]
where \( s \) is the equilibrium strategy profile.

- no agent loses by participating in the mechanism.
- *ex interim* because it holds for *every* possible valuation for agent \( i \), but averages over the possible valuations of the other agents.
Individual-Rationality

Definition (Ex interim individual rationality)
A mechanism is ex interim individual rational when
\( \forall i \forall v_i, \mathbb{E}_{v_{-i}|v_i} v_i(x(s_i(v_i), s_{-i}(v_{-i}))) - p_i(s_i(v_i), s_{-i}(v_{-i})) \geq 0, \) where \( s \) is the equilibrium strategy profile.

- no agent loses by participating in the mechanism.
- ex interim because it holds for every possible valuation for agent \( i \), but averages over the possible valuations of the other agents.

Definition (Ex post individual rationality)
A mechanism is ex post individual rational when
\( \forall i \forall v, v_i(x(s(v))) - p_i(s(v)) \geq 0, \) where \( s \) is the equilibrium strategy profile.
Definition (Tractability)

A mechanism is **tractable** when $\forall \hat{v}$, $\chi(\hat{v})$ and $p(\hat{v})$ can be computed in polynomial time.

- The mechanism is computationally feasible.
Revenue Maximization

We can also add an objective function to our mechanism. One example: revenue maximization.

Definition (Revenue maximization)

A mechanism is revenue maximizing when, among the set of functions $\chi$ and $p$ that satisfy the other constraints, the mechanism selects the $\chi$ and $p$ that maximize $\mathbb{E}_\theta \sum_i p_i(s(\theta))$, where $s(\theta)$ denotes the agents’ equilibrium strategy profile.

- The mechanism designer can choose among mechanisms that satisfy the desired constraints by adding an objective function such as revenue maximization.
The mechanism may not be intended to make money.
Budget balance may be impossible to satisfy.
Set weak budget balance as a constraint and add the following objective.

**Definition (Revenue minimization)**

A quasilinear mechanism is revenue minimizing when, among the set of functions $\chi$ and $p$ that satisfy the other constraints, the mechanism selects the $\chi$ and $p$ that minimize $\max_v \sum_i p_i(s(v))$ in equilibrium, where $s(v)$ denotes the agents’ equilibrium strategy profile.

Note: this considers the worst case over valuations; we could consider average case instead.
Fairness

- Fairness is hard to define. What is fairer:
  - an outcome that fines all agents $100 and makes a choice that all agents hate equally?
  - an outcome that charges all agents $0 and makes a choice that some agents hate and some agents like?
Fairness

- Fairness is hard to define. What is fairer:
  - an outcome that fines all agents $100 and makes a choice that all agents hate equally?
  - an outcome that charges all agents $0 and makes a choice that some agents hate and some agents like?

- Maxmin fairness: make the least-happy agent the happiest.

**Definition (Maxmin fairness)**

A quasilinear mechanism is **maxmin fair** when, among the set of functions $\chi$ and $p$ that satisfy the other constraints, the mechanism selects the $\chi$ and $p$ that maximize

$$
\mathbb{E}_v \left[ \min_{i \in N} v_i(\chi(s(v))) - p_i(s(v)) \right],
$$

where $s(v)$ denotes the agents’ equilibrium strategy profile.
When an efficient mechanism is impossible, we may want to get as close as possible

Minimize the worst-case ratio between optimal social welfare and the social welfare achieved by the given mechanism.

**Definition (Price-of-anarchy minimization)**

A quasilinear mechanism minimizes the price of anarchy when, among the set of functions $\chi$ and $p$ that satisfy the other constraints, the mechanism selects the $\chi$ and $p$ that minimize

$$\max_{v \in V} \max_{x \in X} \frac{\sum_{i \in N} v_i(x)}{\sum_{i \in N} v_i(\chi(s(v)))},$$

where $s(v)$ denotes the agents' equilibrium strategy profile in the worst equilibrium of the mechanism—i.e., the one in which $\sum_{i \in N} v_i(\chi(s(v)))$ is the smallest.
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A positive result

- Recall that in the quasilinear utility setting, a mechanism can be defined as a **choice rule** and a **payment rule**.
- The **Groves mechanism** is a mechanism that satisfies:
  - dominant strategy (truthfulness)
  - efficiency
- In general it’s not:
  - budget balanced
  - individual-rational

...though we’ll see later that there’s some hope for recovering these properties.
The Groves Mechanism

Definition (Groves mechanism)

The Groves mechanism is a direct quasilinear mechanism \((\chi, p)\), where

\[
\chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x)
\]

\[
p_i(\hat{v}) = h_i(\hat{v}_{-i}) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))
\]
The Groves Mechanism

\[ \chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x) \]

\[ p_i(\hat{v}) = h_i(\hat{v}_{-i}) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \]

- The choice rule should not come as a surprise (why not?)
The choice rule should not come as a surprise (why not?) because the mechanism is both truthful and efficient: these properties entail the given choice rule.

\[ \chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x) \]

\[ p_i(\hat{v}) = h_i(\hat{v}_{-i}) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \]
The Groves Mechanism

\[ \chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x) \]

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- The choice rule should not come as a surprise (why not?) because the mechanism is both truthful and efficient: these properties entail the given choice rule.
- So what’s going on with the payment rule?
  - the agent \( i \) must pay some amount \( h_i(\hat{v}_{-i}) \) that doesn't depend on his own declared valuation
  - the agent \( i \) is paid \( \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \), the sum of the others’ valuations for the chosen outcome
Groves Truthfulness

**Theorem**

*Truth telling is a dominant strategy under the Groves mechanism.*

Consider a situation where every agent \( j \) other than \( i \) follows some arbitrary strategy \( \hat{v}_j \). Consider agent \( i \)'s problem of choosing the best strategy \( \hat{v}_i \). As a shorthand, we will write \( \hat{v} = (\hat{v}_{-i}, \hat{v}_i) \). The best strategy for \( i \) is one that solves

\[
\max_{\hat{v}_i} \left( v_i(\chi(\hat{v})) - p(\hat{v}) \right)
\]

Substituting in the payment function from the Groves mechanism, we have

\[
\max_{\hat{v}_i} \left( v_i(\chi(\hat{v})) - h_i(\hat{v}_{-i}) + \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \right)
\]

Since \( h_i(\hat{v}_{-i}) \) does not depend on \( \hat{v}_i \), it is sufficient to solve

\[
\max_{\hat{v}_i} \left( v_i(\chi(\hat{v})) + \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \right).
\]
Groves Truthfulness

\[
\max_{\hat{v}_i} \left( v_i(\chi(\hat{v})) + \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \right).
\]

The only way the declaration \( \hat{v}_i \) influences this maximization is through the choice of \( x \). If possible, \( i \) would like to pick a declaration \( \hat{v}_i \) that will lead the mechanism to pick an \( x \in X \) which solves

\[
\max_x \left( v_i(x) + \sum_{j \neq i} \hat{v}_j(x) \right).
\] (1)

Under the Groves mechanism,

\[
\chi(\hat{v}) = \arg \max_x \left( \sum_i \hat{v}_i(x) \right) = \arg \max_x \left( \hat{v}_i(x) + \sum_{j \neq i} \hat{v}_j(x) \right).
\]

The Groves mechanism will choose \( x \) in a way that solves the maximization problem in Equation (1) when \( i \) declares \( \hat{v}_i = v_i \). Because this argument does not depend in any way on the declarations of the other agents, truth-telling is a dominant strategy for agent \( i \).
Proof intuition

- externalities are internalized
  - agents may be able to change the outcome to another one that they prefer, by changing their declaration
  - however, their utility doesn’t just depend on the outcome—it also depends on their payment
  - since they get paid the (reported) utility of all the other agents under the chosen allocation, they now have an interest in maximizing everyone’s utility rather than just their own
- in general, DS truthful mechanisms have the property that an agent’s payment doesn’t depend on the amount of his declaration, but only on the other agents’ declarations
  - the agent’s declaration is used only to choose the outcome, and to set other agents’ payments
An efficient social choice function \( C : \mathbb{R}^{X^n} \rightarrow X \times \mathbb{R}^n \) can be implemented in dominant strategies for agents with unrestricted quasilinear utilities only if \( p_i(v) = h(v_{-i}) - \sum_{j \neq i} v_j(x(v)) \).

It turns out that the same result also holds for the broader class of Bayes–Nash incentive-compatible efficient mechanisms.