Analyzing Bayesian Games; Social Choice

Lecture 11
Lecture Overview

1. Recap
2. Analyzing Bayesian games
3. Social Choice
4. Voting Paradoxes
Formal Definition

Definition

A stochastic game is a tuple $(Q, N, A_1, \ldots, A_n, P, r_1, \ldots, r_n)$, where

- $Q$ is a finite set of states,
- $N$ is a finite set of $n$ players,
- $A_i$ is a finite set of actions available to player $i$. Let $A = A_1 \times \cdots \times A_n$ be the vector of all players’ actions,
- $P : Q \times A \times Q \to [0, 1]$ is the transition probability function; let $P(q, a, \hat{q})$ be the probability of transitioning from state $s$ to state $\hat{q}$ after joint action $a$,
- $r_i : Q \times A \to \mathbb{R}$ is a real-valued payoff function for player $i$. 

Strategies

- What is a pure strategy?
  - pick an action conditional on every possible history
  - of course, mixtures over these pure strategies are possible too!

- Some interesting restricted classes of strategies:
  - behavioral strategy: \( s_i(h_t, a_{i,j}) \) returns the probability of playing action \( a_{i,j} \) for history \( h_t \).
    - the substantive assumption here is that mixing takes place at each history independently, not once at the beginning of the game
  - Markov strategy: \( s_i \) is a behavioral strategy in which \( s_i(h_t, a_{i,j}) = s_i(h_t', a_{i,j}) \) if \( q_t = q_t' \), where \( q_t \) and \( q_t' \) are the final states of \( h_t \) and \( h_t' \), respectively.
    - for a given time \( t \), the distribution over actions only depends on the current state
  - stationary strategy: \( s_i \) is a Markov strategy in which \( s_i(h_{t_1}, a_{i,j}) = s_i(h_{t_2}', a_{i,j}) \) if \( q_{t_1} = q_{t_2}' \), where \( q_{t_1} \) and \( q_{t_2}' \) are the final states of \( h_{t_1} \) and \( h_{t_2}' \), respectively.
    - no dependence even on \( t \)
Definition 1: Information Sets

- **Bayesian game**: a set of games that differ only in their payoffs, a common prior defined over them, and a partition structure over the games for each agent.

**Definition (Bayesian Game: Information Sets)**

A **Bayesian game** is a tuple \((N, G, P, I)\) where

- \(N\) is a set of agents,
- \(G\) is a set of games with \(N\) agents each such that if \(g, g' \in G\) then for each agent \(i \in N\) the strategy space in \(g\) is identical to the strategy space in \(g'\),
- \(P \in \Pi(G)\) is a common prior over games, where \(\Pi(G)\) is the set of all probability distributions over \(G\), and
- \(I = (I_1, ..., I_N)\) is a set of partitions of \(G\), one for each agent.
### Definition 1: Example

We consider a Bayesian game with two information sets: $I_{1,1}$ and $I_{1,2}$.

#### $I_{2,1}$
- **A**
  - 1 | 0  
  - 0 | 3  
  - $p = 0.2$
- **B**
  - 2 | 1  
  - 1 | 4  
  - $p = 0.1$

#### $I_{2,2}$
- **C**
  - 1 | 3  
  - 2 | 1  
  - $p = 0.1$
- **D**
  - 1 | 0  
  - 4 | 0  
  - $p = 0$

#### $I_{1,2}$
- **E**
  - 3 | 2  
  - 0 | 3  
  - $p = 0.1$
- **F**
  - 2 | 4  
  - 1 | 1  
  - $p = 0.1$
- **G**
  - 0 | 1  
  - 3 | 1  
  - $p = 0.25$
- **H**
  - 2 | 0  
  - 3 | 4  
  - $p = 0.15$
Definition 2: Extensive Form with Chance Moves

- Add an agent, “Nature,” who follows a commonly known mixed strategy.
- Thus, reduce Bayesian games to extensive form games of imperfect information.
- This definition is cumbersome for the same reason that IIEF is a cumbersome way of representing matrix games like Prisoner’s dilemma
  - however, it makes sense when the agents really do move sequentially, and at least occasionally observe each other’s actions.
Definition 2: Example

Figure 6.7 A Bayesian game

receive individual signals about Nature's choice, and these are captured by their information sets, in a standard way. The agents have no additional information; in particular, the information sets capture the fact that agents make their choices without knowing the choices of others. Thus, we have reduced games of incomplete information to games of imperfect information, albeit ones with chance moves. These chance moves of Nature require minor adjustments of existing definitions, replacing payoffs by their expectations, given Nature's moves.

For example, the Bayesian game of Figure 6.7 can be represented in extensive form as depicted in Figure 6.8.

Figure 6.8 The Bayesian game from Figure 6.7 in extensive form

Although this second definition of Bayesian games can be more initially intuitive than our first definition, it can also be more cumbersome to work with. This is because the special structure of this extensive form means that we do not have to agonize over the refinements of Nash equilibrium; since agents have no information about prior choices made other than by Nature, all Nash equilibria are also sequential equilibria.

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Definition 3: Epistemic Types

- Directly represent uncertainty over utility function using the notion of epistemic type.

Definition

A Bayesian game is a tuple \((N, A, \Theta, p, u)\) where

- \(N\) is a set of agents,
- \(A = (A_1, \ldots, A_n)\), where \(A_i\) is the set of actions available to player \(i\),
- \(\Theta = (\Theta_1, \ldots, \Theta_n)\), where \(\Theta_i\) is the type space of player \(i\),
- \(p : \Theta \rightarrow [0, 1]\) is the common prior over types,
- \(u = (u_1, \ldots, u_n)\), where \(u_i : A \times \Theta \rightarrow \mathbb{R}\) is the utility function for player \(i\).
### Definition 3: Example

#### Payoff Matrices

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#### Strategy Profiles

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Lecture Overview

1 Recap

2 Analyzing Bayesian games

3 Social Choice

4 Voting Paradoxes
Strategies

- **Pure strategy**: $s_i : \Theta_i \rightarrow A_i$
  - a mapping from every type agent $i$ could have to the action he would play if he had that type.

- **Mixed strategy**: $s_i : \Theta_i \rightarrow \Pi(A_i)$
  - a mapping from $i$’s type to a probability distribution over his action choices.

- $s_j(a_j|\theta_j)$
  - the probability under mixed strategy $s_j$ that agent $j$ plays action $a_j$, given that $j$’s type is $\theta_j$. 
Expected Utility

Three meaningful notions of expected utility:

- **ex-ante**
  - the agent knows nothing about anyone’s actual type;

- **ex-interim**
  - an agent knows his own type but not the types of the other agents;

- **ex-post**
  - the agent knows all agents’ types.
Ex-interim expected utility

Definition (Ex-interim expected utility)

Agent $i$’s *ex-interim* expected utility in a Bayesian game $(N, A, \Theta, p, u)$, where $i$’s type is $\theta_i$ and where the agents’ strategies are given by the mixed strategy profile $s$, is defined as

$$EU_i(s|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) \sum_{a \in A} \left( \prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a, \theta_{-i}, \theta_i).$$

- $i$ must consider every $\theta_{-i}$ and every $a$ in order to evaluate $u_i(a, \theta_i, \theta_{-i})$.
- $i$ must weight this utility value by:
  - the probability that $a$ would be realized given all players’ mixed strategies and types;
  - the probability that the other players’ types would be $\theta_{-i}$ given that his own type is $\theta_i$. 

**Ex-ante expected utility**

**Definition (Ex-ante expected utility)**

Agent $i$’s *ex-ante expected utility* in a Bayesian game $(N, A, \Theta, p, u)$, where the agents’ strategies are given by the mixed strategy profile $s$, is defined as

$$EU_i(s) = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s|\theta_i)$$

or equivalently as

$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) \sum_{a \in A} \left( \prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a, \theta).$$
**Ex-post** expected utility

**Definition (Ex-post expected utility)**

Agent $i$’s *ex-post* expected utility in a Bayesian game $(N, A, \Theta, p, u)$, where the agents’ strategies are given by $s$ and the agent’ types are given by $\theta$, is defined as

$$E U_i(s, \theta) = \sum_{a \in A} \left( \prod_{j \in N} s_j(a_j | \theta_j) \right) u_i(a, \theta).$$

- The only uncertainty here concerns the other agents’ mixed strategies, since $i$ knows everyone’s type.
Best response

Definition (Best response in a Bayesian game)

The set of agent $i$’s best responses to mixed strategy profile $s_{-i}$ are given by

$$BR_i(s_{-i}) = \arg \max_{s_i' \in S} EU_i(s_i', s_{-i}).$$

- It may seem odd that $BR$ is calculated based on $i$’s ex-ante expected utility.
- However, write $EU_i(s)$ as $\sum_{\theta_i \in \Theta} p(\theta_i) EU_i(s | \theta_i)$ and observe that $EU_i(s_i', s_{-i} | \theta_i)$ does not depend on strategies that $i$ would play if his type were not $\theta_i$.
- Thus, we are in fact performing independent maximization of $i$’s ex-interim expected utility conditioned on each type that he could have.
Nash equilibrium

**Definition (Bayes-Nash equilibrium)**

A Bayes-Nash equilibrium is a mixed strategy profile $s$ that satisfies
\[ \forall i \ s_i \in BR_i(s_{-i}). \]

- we can also construct an induced normal form for Bayesian games
- the numbers in the cells will correspond to \textit{ex-ante} expected utilities
- however as argued above, as long as the strategy space is unchanged, best responses don’t change between the \textit{ex-ante} and \textit{ex-interim} cases.
**ex-post Equilibrium**

**Definition (ex-post equilibrium)**

A *ex-post equilibrium* is a mixed strategy profile $s$ that satisfies

$$\forall \theta, \forall i, \ s_i \in \arg \max_{s_i' \in S_i} EU_i(s_i', s_{-i}, \theta).$$

- somewhat similar to dominant strategy, but not quite
  - EP: agents do not need to have accurate beliefs about the type distribution
  - DS: agents do not need to have accurate beliefs about others’ strategies
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Introduction

Our setting now:

- a set of outcomes
- agents have preferences across them
- for the moment, we won’t consider incentive issues:
  - center knows agents’ preferences, or they declare truthfully
- the goal: a social choice function: a mapping from everyone’s preferences to a particular outcome, which is enforced
  - how to pick such functions with desirable properties?
Definition (Social choice function)
Assume a set of agents $N = \{1, 2, \ldots, n\}$, and a set of outcomes (or alternatives, or candidates) $O$. Let $L_-$ be the set of non-strict total orders on $O$. A social choice function (over $N$ and $O$) is a function $C : L_-^n \mapsto O$.

Definition (Social welfare function)
Let $N, O, L_-$ be as above. A social welfare function (over $N$ and $O$) is a function $W : L_-^n \mapsto L_-$.
Non-Ranking Voting Schemes

- **Plurality**
  - pick the outcome which is preferred by the most people

- **Cumulative voting**
  - distribute e.g., 5 votes each
  - possible to vote for the same outcome multiple times

- **Approval voting**
  - accept as many outcomes as you “like”
Ranking Voting Schemes

- **Plurality with elimination** ("instant runoff")
  - everyone selects their favorite outcome
  - the outcome with the fewest votes is eliminated
  - repeat until one outcome remains

- **Borda**
  - assign each outcome a number.
  - The most preferred outcome gets a score of $n - 1$, the next most preferred gets $n - 2$, down to the $n^{th}$ outcome which gets 0.
  - Then sum the numbers for each outcome, and choose the one that has the highest score

- **Pairwise elimination**
  - in advance, decide a schedule for the order in which pairs will be compared.
  - given two outcomes, have everyone determine the one that they prefer
  - eliminate the outcome that was not preferred, and continue with the schedule
Condorcet Condition

- If there is a candidate who is preferred to every other candidate in pairwise runoffs, that candidate should be the winner.
- While the Condorcet condition is considered an important property for a voting system to satisfy, there is not always a Condorcet winner.
- Sometimes, there’s a cycle where A defeats B, B defeats C, and C defeats A in their pairwise runoffs.
Condorcet example

499 agents: $A \succ B \succ C$

3 agents: $B \succ C \succ A$

498 agents: $C \succ B \succ A$

• What is the Condorcet winner?
Condorcet example

499 agents: $A \succ B \succ C$

3 agents: $B \succ C \succ A$

498 agents: $C \succ B \succ A$

- What is the Condorcet winner? $B$
Condorcet example

499 agents: \( A \succ B \succ C \)
3 agents: \( B \succ C \succ A \)
498 agents: \( C \succ B \succ A \)

- What is the Condorcet winner? \( B \)
- What would win under plurality voting?
Condorcet example

499 agents: \( A \succ B \succ C \)
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- What is the Condorcet winner? \( B \)
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- What would win under plurality voting? \( A \)
- What would win under plurality with elimination?
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499 agents: $A \succ B \succ C$

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- What is the Condorcet winner? $B$
- What would win under plurality voting? $A$
- What would win under plurality with elimination? $C$
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Sensitivity to Losing Candidate

35 agents: $A \succ C \succ B$
33 agents: $B \succ A \succ C$
32 agents: $C \succ B \succ A$

What candidate wins under plurality voting?
Sensitivity to Losing Candidate

35 agents: $A \succ C \succ B$
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- What candidate wins under plurality voting? \( A \)
- What candidate wins under Borda voting?
Sensitivity to Losing Candidate

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Sensitivity to Losing Candidate

35 agents: $A \succ C \succ B$
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- What candidate wins under plurality voting? $A$
- What candidate wins under Borda voting? $A$
- Now consider dropping $C$. Now what happens under both Borda and plurality?
Sensitivity to Losing Candidate

- 35 agents: $A \succ C \succ B$
- 33 agents: $B \succ A \succ C$
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- What candidate wins under plurality voting? $A$
- What candidate wins under Borda voting? $A$
Sensitivity to Agenda Setter

35 agents: \( A \succ C \succ B \)
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- Who wins pairwise elimination, with the ordering \( A, B, C \)?
Sensitivity to Agenda Setter

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Sensitivity to Agenda Setter

35 agents: $A \succ C \succ B$
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- Who wins pairwise elimination, with the ordering $A, B, C$? $C$
- Who wins with the ordering $A, C, B$? $B$
- Who wins with the ordering $B, C, A$? $A$
Another Pairwise Elimination Problem

1 agent:  \( B \succ D \succ C \succ A \)
1 agent:  \( A \succ B \succ D \succ C \)
1 agent:  \( C \succ A \succ B \succ D \)

Who wins under pairwise elimination with the ordering \( A, B, C, D \)?

What is the problem with this?

All of the agents prefer \( B \) to \( D \) — the selected candidate is Pareto-dominated!
Another Pairwise Elimination Problem

1 agent: $B \succ D \succ C \succ A$
1 agent: $A \succ B \succ D \succ C$
1 agent: $C \succ A \succ B \succ D$

Who wins under pairwise elimination with the ordering $A, B, C, D$? $D$. 

What is the problem with this?

All of the agents prefer $B$ to $D$ — the selected candidate is Pareto-dominated!
Another Pairwise Elimination Problem

1 agent: \( B \succ D \succ C \succ A \)
1 agent: \( A \succ B \succ D \succ C \)
1 agent: \( C \succ A \succ B \succ D \)

- Who wins under pairwise elimination with the ordering \( A, B, C, D? \) \( D \).
- What is the problem with this?
Another Pairwise Elimination Problem

1 agent: \( B \succ D \succ C \succ A \)
1 agent: \( A \succ B \succ D \succ C \)
1 agent: \( C \succ A \succ B \succ D \)

- Who wins under pairwise elimination with the ordering \( A, B, C, D? \) \( D \).
- What is the problem with this?
  - all of the agents prefer \( B \) to \( D \)—the selected candidate is Pareto-dominated!