Stochastic Games and Bayesian Games

CPSC 532l Lecture 10
Lecture Overview

1. Recap
2. Stochastic Games
3. Bayesian Games
4. Analyzing Bayesian games
Finitely Repeated Games

- Everything is straightforward if we repeat a game a finite number of times.
- We can write the whole thing as an extensive-form game with imperfect information.
  - At each round players don’t know what the others have done; afterwards they do.
  - Overall payoff function is additive: sum of payoffs in stage games.
Consider an infinitely repeated game in extensive form:
- an infinite tree!

Thus, payoffs cannot be attached to terminal nodes, nor can they be defined as the sum of the payoffs in the stage games (which in general will be infinite).

**Definition**

Given an infinite sequence of payoffs $r_1, r_2, \ldots$ for player $i$, the average reward of $i$ is

$$\lim_{k \to \infty} \sum_{j=1}^{k} \frac{r_j}{k}.$$
Nash Equilibria

- With an infinite number of equilibria, what can we say about Nash equilibria?
  - we won't be able to construct an induced normal form and then appeal to Nash’s theorem to say that an equilibrium exists
  - Nash’s theorem only applies to finite games
- Furthermore, with an infinite number of strategies, there could be an infinite number of pure-strategy equilibria!
- It turns out we can characterize a set of payoffs that are achievable under equilibrium, without having to enumerate the equilibria.
Definitions

Consider any $n$-player game $G = (N, A, u)$ and any payoff vector $r = (r_1, r_2, \ldots, r_n)$.

Let $v_i = \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_{-i}, s_i)$.

$i$’s minmax value: the amount of utility $i$ can get when $-i$ play a minmax strategy against him.

**Definition**

A payoff profile $r$ is **enforceable** if $r_i \geq v_i$.

**Definition**

A payoff profile $r$ is **feasible** if there exist rational, non-negative values $\alpha_a$ such that for all $i$, we can express $r_i$ as $\sum_{a \in A} \alpha_u_i(a)$, with $\sum_{a \in A} \alpha_a = 1$.

A payoff profile is feasible if it is a convex, rational combination of the outcomes in $G$. 
Folk Theorem

Consider any \( n \)-player game \( G \) and any payoff vector \((r_1, r_2, \ldots, r_n)\).

1. If \( r \) is the payoff in any Nash equilibrium of the infinitely repeated \( G \) with average rewards, then for each player \( i \), \( r_i \) is enforceable.

2. If \( r \) is both feasible and enforceable, then \( r \) is the payoff in some Nash equilibrium of the infinitely repeated \( G \) with average rewards.
Folk Theorem (Part 1)

Payoff in Nash $\rightarrow$ enforceable

**Part 1:** Suppose $r$ is not enforceable, i.e. $r_i < v_i$ for some $i$. Then consider a deviation of this player $i$ to $b_i(s_{-i}(h))$ for any history $h$ of the repeated game, where $b_i$ is any best-response action in the stage game and $s_{-i}(h)$ is the equilibrium strategy of other players given the current history $h$. By definition of a minmax strategy, player $i$ will receive a payoff of at least $v_i$ in every stage game if he adopts this strategy, and so $i$’s average reward is also at least $v_i$. Thus $i$ cannot receive the payoff $r_i < v_i$ in any Nash equilibrium.
Folk Theorem (Part 2)

Feasible and enforceable → Nash

**Part 2:** Since $r$ is a feasible payoff profile, we can write it as
$$r_i = \sum_{a \in A} \left( \frac{\beta_a}{\gamma} \right) u_i(a),$$
where $\beta_a$ and $\gamma$ are non-negative integers.\(^1\)

Since the combination was convex, we have $\gamma = \sum_{a \in A} \beta_a$.

We’re going to construct a strategy profile that will cycle through all outcomes $a \in A$ of $G$ with cycles of length $\gamma$, each cycle repeating action $a$ exactly $\beta_a$ times. Let $(a^t)$ be such a sequence of outcomes. Let’s define a strategy $s_i$ of player $i$ to be a trigger version of playing $(a^t)$: if nobody deviates, then $s_i$ plays $a_i^t$ in period $t$. However, if there was a period $t'$ in which some player $j \neq i$ deviated, then $s_i$ will play $(p_{-j})_i$, where $(p_{-j})$ is a solution to the minimization problem in the definition of $v_j$.

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\(^1\)Recall that $\alpha_a$ were required to be rational. So we can take $\gamma$ to be their common denominator.
Folk Theorem (Part 2)

Feasible and enforceable $\rightarrow$ Nash

First observe that if everybody plays according to $s_i$, then, by construction, player $i$ receives average payoff of $r_i$ (look at averages over periods of length $\gamma$). Second, this strategy profile is a Nash equilibrium. Suppose everybody plays according to $s_i$, and player $j$ deviates at some point. Then, forever after, player $j$ will receive his min max payoff $v_j \leq r_j$, rendering the deviation unprofitable.
<table>
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Lecture Overview

1. Recap
2. Stochastic Games
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What if we didn’t always repeat back to the same stage game?

A stochastic game is a generalization of repeated games
- agents repeatedly play games from a set of normal-form games
- the game played at any iteration depends on the previous game played and on the actions taken by all agents in that game

A stochastic game is a generalized Markov decision process
- there are multiple players
- one reward function for each agent
- the state transition function and reward functions depend on the action choices of both players
A stochastic game is a tuple \((Q, N, A, P, R)\), where

- \(Q\) is a finite set of states,
- \(N\) is a finite set of \(n\) players,
- \(A = A_1 \times \cdots \times A_n\), where \(A_i\) is a finite set of actions available to player \(i\),
- \(P : Q \times A \times Q \mapsto [0, 1]\) is the transition probability function; \(P(q, a, \hat{q})\) is the probability of transitioning from state \(s\) to state \(\hat{q}\) after joint action \(a\), and
- \(R = r_1, \ldots, r_n\), where \(r_i : Q \times A \mapsto \mathbb{R}\) is a real-valued payoff function for player \(i\).
Remarks

- This assumes strategy space is the same in all games
  - otherwise just more notation
- Again we can have average or discounted payoffs.
- Interesting special cases:
  - zero-sum stochastic game
  - single-controller stochastic game
    - transitions (but not payoffs) depend on only one agent
Strategies

- What is a pure strategy?
Strategies

- What is a pure strategy?
  - pick an action conditional on every possible history
  - of course, mixtures over these pure strategies are possible too!

- Some interesting restricted classes of strategies:
  - **behavioral strategy**: \( s_i(h_t, a_{ij}) \) returns the probability of playing action \( a_{ij} \) for history \( h_t \).
    - the substantive assumption here is that mixing takes place at each history independently, not once at the beginning of the game
  - **Markov strategy**: \( s_i \) is a behavioral strategy in which \( s_i(h_t, a_{ij}) = s_i(h'_t, a_{ij}) \) if \( q_t = q'_t \), where \( q_t \) and \( q'_t \) are the final states of \( h_t \) and \( h'_t \), respectively.
    - for a given time \( t \), the distribution over actions only depends on the current state
  - **stationary strategy**: \( s_i \) is a Markov strategy in which \( s_i(h_{t1}, a_{ij}) = s_i(h'_{t2}, a_{ij}) \) if \( q_{t1} = q'_{t2} \), where \( q_{t1} \) and \( q'_{t2} \) are the final states of \( h_{t1} \) and \( h'_{t2} \), respectively.
    - no dependence even on \( t \)
Equilibrium (discounted rewards)

- **Markov perfect equilibrium:**
  - a strategy profile consisting of only Markov strategies that is a Nash equilibrium regardless of the starting state
  - analogous to subgame-perfect equilibrium

**Theorem**

*Every* $n$*-player, general sum, discounted reward stochastic game has a Markov perfect equilibrium.*
Equilibrium (average rewards)

- **Irreducible stochastic game:**
  - every strategy profile gives rise to an irreducible Markov chain over the set of games
    - irreducible Markov chain: possible to get from every state to every other state
  - during the (infinite) execution of the stochastic game, each stage game is guaranteed to be played infinitely often—for any strategy profile
  - without this condition, limit of the mean payoffs may not be defined

**Theorem**

For every 2-player, general sum, average reward, irreducible stochastic game has a Nash equilibrium.
A folk theorem

Theorem

For every 2-player, general sum, irreducible stochastic game, and every feasible outcome with a payoff vector $r$ that provides to each player at least his minmax value, there exists a Nash equilibrium with a payoff vector $r$. This is true for games with average rewards, as well as games with large enough discount factors (i.e. with players that are sufficiently patient).
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Fun Game

Choose a phone number none of your neighbours knows; consider it to be ABC-DEFG
Fun Game

- Choose a phone number none of your neighbours knows; consider it to be ABC-DEFG
  - take “DE” as your valuation
  - play a first-price auction with three neighbours, where your utility is your valuation minus the amount you pay

Questions:
- What is the role of uncertainty here?
- Can we model this uncertainty using an imperfect information extensive form game?
  - Imperfect info means not knowing what node you're in in the info set.
  - Here we're not sure what game is being played (though if we allow a move by nature, we can do it.)
Choose a phone number none of your neighbours knows; consider it to be ABC-DEFG
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- play a first-price auction with three neighbours, where your utility is your valuation minus the amount you pay
- now play the auction again, same neighbours, same valuation

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- now play the auction again, same neighbours, same valuation
- now play again, with “FG” as your valuation
Fun Game

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Introduction

So far, we’ve assumed that all players know what game is being played. Everyone knows:

- the number of players
- the actions available to each player
- the payoff associated with each action vector

Why is this true in imperfect information games?

We’ll assume:

1. All possible games have the same number of agents and the same strategy space for each agent; they differ only in their payoffs.
2. The beliefs of the different agents are posteriors, obtained by conditioning a common prior on individual private signals.
Definition 1: Information Sets

- **Bayesian game**: a set of games that differ only in their payoffs, a common prior defined over them, and a partition structure over the games for each agent.

**Definition (Bayesian Game: Information Sets)**

A Bayesian game is a tuple \((N, G, P, I)\) where

- \(N\) is a set of agents,
- \(G\) is a set of games with \(N\) agents each such that if \(g, g' \in G\) then for each agent \(i \in N\) the strategy space in \(g\) is identical to the strategy space in \(g'\),
- \(P \in \Pi(G)\) is a common prior over games, where \(\Pi(G)\) is the set of all probability distributions over \(G\), and
- \(I = (I_1, ..., I_N)\) is a set of partitions of \(G\), one for each agent.
## Definition 1: Example

Stochastic Games and Bayesian Games

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### Figure 6.7

A Bayesian game

Receive individual signals about Nature’s choice, and these are captured by their information sets, in a standard way. The agents have no additional information; in particular, the information sets capture the fact that agents make their choices without knowing the choices of others. Thus, we have reduced games of incomplete information to games of imperfect information, albeit ones with chance moves. These chance moves of Nature require minor adjustments of existing definitions, replacing payoffs by their expectations, given Nature’s moves.

For example, the Bayesian game of Figure 6.7 can be represented in extensive form as depicted in Figure 6.8.

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*Note that the special structure of this extensive form means that we do not have to agonize over the refinements of Nash equilibrium; since agents have no information about prior choices made other than by nature, all Nash equilibria are also sequential equilibria.*

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Add an agent, “Nature,” who follows a commonly known mixed strategy.

Thus, reduce Bayesian games to extensive form games of imperfect information.

This definition is cumbersome for the same reason that IIEF is a cumbersome way of representing matrix games like Prisoner’s dilemma

However, it makes sense when the agents really do move sequentially, and at least occasionally observe each other’s actions.
Recap Stochastic Games Bayesian Games Analyzing Bayesian games

Definition 2: Example

Figure 6.7 A Bayesian game

receive individual signals about Nature’s choice, and these are captured by their information sets, in a standard way. The agents have no additional information; in particular, the information sets capture the fact that agents make their choices without knowing the choices of others. Thus, we have reduced games of incomplete information to games of imperfect information, albeit ones with chance moves. These chance moves of Nature require minor adjustments of existing definitions, replacing payoffs by their expectations, given Nature’s moves.

For example, the Bayesian game of Figure 6.7 can be represented in extensive form as depicted in Figure 6.8.

Although this second definition of Bayesian games can be more initially intuitive than our first definition, it can also be more cumbersome to work with. This is because the special structure of this extensive form means that we do not have to agonize over the refinements of Nash equilibrium; since agents have no information about prior choices made other than by nature, all Nash equilibria are also sequential equilibria.

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Definition 3: Epistemic Types

- Directly represent uncertainty over utility function using the notion of epistemic type.

**Definition**

A Bayesian game is a tuple \((N, A, \Theta, p, u)\) where

- \(N\) is a set of agents,
- \(A = (A_1, \ldots, A_n)\), where \(A_i\) is the set of actions available to player \(i\),
- \(\Theta = (\Theta_1, \ldots, \Theta_n)\), where \(\Theta_i\) is the type space of player \(i\),
- \(p : \Theta \rightarrow [0, 1]\) is the common prior over types,
- \(u = (u_1, \ldots, u_n)\), where \(u_i : A \times \Theta \rightarrow \mathbb{R}\) is the utility function for player \(i\).
Definition 3: Example

Recap

Stochastic Games

Bayesian Games

Analyzing Bayesian games

- Expected utility
- 6.3.2 Analyzing Bayesian games

Expressed in terms of the first two Bayesian game definitions, of course; we leave this
now that we have defined Bayesian games we must explain how to reason about them.
Thus, we have reduced games of incomplete information to
mixed strategy

mixed strategy

Note that the special structure of this extensive form means that we do not have to agonize over the
of Nature require minor adjustments of existing definitions, replacing payoffs by their
as depicted in Figure 6.8.

- Stochastic Games
- Bayesian Games

The first task is to define an agent’s strategy space in a Bayesian game. Recall
the choices of others. Thus, we have reduced games of incomplete information to

For example, the Bayesian game of Figure 6.7 can be represented in extensive form

from information sets to actions. The definition is similar in Bayesian games: a pure

that in an imperfect-information extensive-form game a pure strategy was a mapping

The second considers the setting in which an agent knows his own type but not the types of
the other agents, and in the third case the agent knows all agents’ types.

discusses the situation in which the agent knows nothing about agents’ actual types, the

Although this second definition of Bayesian games can be more initially intuitive

the choices of others. Thus, we have reduced games of incomplete information to

Note that the special structure of this extensive form means that we do not have to agonize over the

... made other than by

nature, all Nash equilibria are also sequential equilibria.

Recap Stochastic Games Bayesian Games Analyzing Bayesian games
Lecture Overview

1 Recap

2 Stochastic Games

3 Bayesian Games

4 Analyzing Bayesian games
Strategies

- **Pure strategy**: \( s_i : \Theta_i \rightarrow A_i \)
  - a mapping from every type agent \( i \) could have to the action he would play if he had that type.

- **Mixed strategy**: \( s_i : \Theta_i \rightarrow \Pi(A_i) \)
  - a mapping from \( i \)'s type to a probability distribution over his action choices.

- \( s_j(a_j|\theta_j) \)
  - the probability under mixed strategy \( s_j \) that agent \( j \) plays action \( a_j \), given that \( j \)'s type is \( \theta_j \).
Expected Utility

Three meaningful notions of expected utility:

- **ex-ante**
  - the agent knows nothing about anyone’s actual type;

- **ex-interim**
  - an agent knows his own type but not the types of the other agents;

- **ex-post**
  - the agent knows all agents’ types.
**Ex-interim expected utility**

**Definition (Ex-interim expected utility)**

Agent $i$’s *ex-interim expected utility* in a Bayesian game $(N, A, \Theta, p, u)$, where $i$’s type is $\theta_i$ and where the agents’ strategies are given by the mixed strategy profile $s$, is defined as

$$EU_i(s|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) \sum_{a \in A} \left( \prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a, \theta_{-i}, \theta_i).$$

- $i$ must consider every $\theta_{-i}$ and every $a$ in order to evaluate $u_i(a, \theta_i, \theta_{-i}).$
- $i$ must weight this utility value by:
  - the probability that $a$ would be realized given all players’ mixed strategies and types;
  - the probability that the other players’ types would be $\theta_{-i}$ given that his own type is $\theta_i.$

Recap

Stochastic Games

Bayesian Games

Analyzing Bayesian games
Ex-ante expected utility

Definition (Ex-ante expected utility)

Agent $i$’s ex-ante expected utility in a Bayesian game $(N, A, \Theta, p, u)$, where the agents’ strategies are given by the mixed strategy profile $s$, is defined as

$$EU_i(s) = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s|\theta_i)$$

or equivalently as

$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) \sum_{a \in A} \left( \prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a, \theta).$$
**Ex-post expected utility**

**Definition (Ex-post expected utility)**

Agent $i$’s *ex-post expected utility* in a Bayesian game $(N, A, \Theta, p, u)$, where the agents’ strategies are given by $s$ and the agent’ types are given by $\theta$, is defined as

$$EU_i(s, \theta) = \sum_{a \in A} \left( \prod_{j \in N} s_j(a_j | \theta_j) \right) u_i(a, \theta).$$

- The only uncertainty here concerns the other agents’ mixed strategies, since $i$ knows everyone’s type.
Best response

Definition (Best response in a Bayesian game)

The set of agent $i$'s best responses to mixed strategy profile $s_{-i}$ are given by

$$BR_i(s_{-i}) = \arg \max_{s'_i \in S_i} EU_i(s'_i, s_{-i}).$$

- It may seem odd that $BR$ is calculated based on $i$'s ex-ante expected utility.

- However, write $EU_i(s)$ as $\sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s|\theta_i)$ and observe that $EU_i(s'_i, s_{-i}|\theta_i)$ does not depend on strategies that $i$ would play if his type were not $\theta_i$.

- Thus, we are in fact performing independent maximization of $i$'s ex-interim expected utility conditioned on each type that he could have.
Definition (Bayes-Nash equilibrium)

A Bayes-Nash equilibrium is a mixed strategy profile $s$ that satisfies
$$\forall i \ s_i \in BR_i(s_{-i}).$$

- we can also construct an induced normal form for Bayesian games
- the numbers in the cells will correspond to \textit{ex-ante} expected utilities
  - however as argued above, as long as the strategy space is unchanged, best responses don’t change between the \textit{ex-ante} and \textit{ex-interim} cases.
**ex-post** Equilibrium

**Definition (ex-post equilibrium)**

A *ex-post* Bayes-Nash equilibrium is a mixed strategy profile $s$ that satisfies $\forall \theta, \forall i, s_i \in \arg \max_{s_i' \in S_i} EU_i(s_i', s_{-i}, \theta)$.

- somewhat similar to dominant strategy, but not quite
  - EP: agents do not need to have accurate beliefs about the type distribution
  - DS: agents do not need to have accurate beliefs about others’ strategies