Power in Voting Games and Canadian Politics

Chris Nicola

December 27, 2006

Abstract

In this work we examine power measures used in the analysis of voting games to quantify power. We consider both weighted and non-weighted voting games giving simple examples of each. The power indices considered are the Shapley-Shubik [8] index, the Banzhoff index [1], the Penrose [7] index and Coleman [2] indices. We apply them to specific examples taken from recent Canadian parliaments and to the work from [6], where the proposed Canadian constitutional amendment scheme from 1971 was examined using the Shapely-Shubik index. Results illustrate both the difficulties in quantifying power in real political systems as well the insight provided by these power indices.

1 Introduction

Voting and politics is an interesting subject for the application of game-theory. Political power can be hard to quantify in a mathematical sense, since it can be difficult to capture the socio-political aspects of real politics. For example, though we may try to define the optimal coalition of parties mathematically, the reality of politics means that parties involved may be ideologically opposed to each other, thus such a coalition is unlikely to form. I would be quite difficulty to measure ideology, though some effort has been made to use it in the modeling optimal coalitions for parliament in [3]. In this work however we show that simple indices for power can be defined by simply assuming all coalitions are equal. We will see that despite this simplifying assumption these indices provide useful insight into political power. Through applying them to real examples from the Canadian government we see how these power measures can be useful in analyzing real voting games.

We begin in Sec. 2 by defining the concepts of both weighted and non-weighted voting games. Then we define the power measures considered in our analysis. In in Sec. 3 the power indices are applied to results from recent Canadian federal elections and we also reconsider results from [6] where a proposed Canadian constitutional amendment scheme from 1971 is examined (an example of a non-weighted voting scheme). To compare, we compute new results using the Banzhoff index and compare this with the actual formula used for amendments to the Canadian constitution. Finally, in Sec. 4 we conclude by reviewing the results and what they demonstrate about what power means in voting games.

2 Voting Games and Measuring Power

2.1 What is a voting game?

In a voting game agents, or groups of agents voting as a block, cast votes indicating their preference for a particular proposition. Only two outcomes need to be considered for a vote, either it passes or it fails. Generally, agents will need to form coalitions to ensure that a vote passes. As mentioned before there may also be external factors affecting the desirability of forming coalitions with certain other agents such as political ideology.

One of the simplest voting games considered is the weighted voting game. Each agent is assigned a weight and the game is defined by it's quota. A coalition can pass a vote if the sum of their weights is greater than the quota.

> Definition: A weighted voting game involving agents $i = 1, \ldots, n$ is defined as,
>
> $$[Q : w_1, w_2, \ldots, w_n],$$
>
> where w_i is the weight assigned to agent i and Q is the minimum quota need to pass a vote. If the parties in coalition C vote for a proposition the vote passes if and only if
>
> $$\sum_{w_i \in C} w_i \geq Q.$$

How do we know if a specific voting scheme is a weighted voting game? For very simple cases it will often be simple to put them into weighted form and show directly they are weighted. For complex games with a large number of players this will not generally be simple. One way to determine if a game is *not* a weighted voting game is to show that the following holds[1]:

> For two winning coalitions C_1 and C_2 where there exist agents i and j s.t. $i \neq j$ and $i \in C_1, j \in C_2$ but $i \notin C_1, j \notin C_2$ we can say that the game not weighted if we can switch i and j in these two coalitions and have it that both are now losing coalitions.

[1] Taken from [5]

In a weighted voting game we must have either that one of the coalitions has greater weight after the swap than before or they are both unchanged. If we can find an example where both are no longer winning after the swap then the game must not be a weighted voting game. We will see that the Canadian constitutional amendment scheme proposed in 1971 is exactly such a game.

2.2 How do we measure power?

It can be easily shown that the relative weightings of agents in a weighted voting game are not, in general, an effective measure of power. For example, in the game $[100 : 99, 1]$, both agents are required for passing any vote. Despite very different weightings their power to vote is identical. Furthermore, if we want to measure power in a non-weighted game weightings are of no use. This leads us to look to other measures of voting power.

2.2.1 Shapley-Shubik Power

The Shapley-Shubik index (SSI) [8] defines power as a percentage of the number of vote orderings in which an agent is pivotal.

> Definition: An agents vote is a swing or pivotal vote in a coalition if their vote can change the outcome of the vote from pass to fail.

For n agents there are $n!$ unique orderings and one agent per ordering who is the first pivotal vote. The number of swings for each agent is divided by the number of swings for all of the agents to compute the SSI. This is to measure how many coalitions an agent is the swing vote in. In some cases an agent may not be pivotal in any vote so that even though he has strictly positive weight he has no power. Alternatively an agent may be pivotal in every coalition giving him an effective veto.

2.2.2 Banzhoff Power

The Banzhoff power index or normalized Banzhoff index (NBI) was developed in [1]. The difference between this and the SSI is how coalitions are considered. Ordering is not important as this examines each of the 2^n possible coalitions and counts how many of these coalitions an agent is pivotal in. The number of swings for each agent is again divided by the number of swings for all agents to compute the NBI.

The difference between NBI and SSI is that SSI only considers minimal voting coalitions, since for each ordering we only consider the first minimal coalition. While it may seem that proportionally these two approaches should be quite similar (and they often are) they will sometimes result in significantly different results

2.2.3 Other Power Indices

There are three modified Banzhoff power indices which we also examine. The Penrose index or absolute Banzhoff index (ABI) [7] divides the number of swings for an agent only by the number of swings for all the other agents.

Two other indices introduced in [2] are the power to prevent action (PPA) and power to initiate action (PIA). The PPA is the number of swings for an agent divided by the number of outcomes that lead to a decision. The purpose of the PPA is to measure the power of an agent to block decisions. The PIA is the number of swings for an agent divided by the number of outcomes that do not lead to a decision. This measures the power of an agent to get a decision made.

It is worth mentioning that for large voting bodies all these indices are very hard to compute. Thus, it is common to use a stochastic method such as a Monte-Carlo method and instead compute them approximately.

3 Analysis of Canadian Political Examples

The last two federal elections in Canada resulted in minority governments which are interesting applications for the measures defined above. Below we examine these elections, the power of each province by seats in federal parliament and the power of each province in amending the Canadian constitution using the indices defined above.

3.1 Canadian Federal Parliament[2]

First, lets consider the division of power in federal parliament among the provinces and territories.

Province	Pop. %	Seats	SSI	NBI	ABI	PIA	PPA
BC	13.2%	36	13.4%	14.18%	25.66%	25.8%	25.5%
AB	10.4%	28	9%	10.43%	18.87%	19%	18.7%
SK	3%	14	4.3%	4.38%	7.94%	8%	7.9%
MB	3.6%	14	4.3%	4.38%	7.94%	8%	7.9%
ON	38.9%	106	39.9%	39.22%	70.97%	71.5%	70.4%
QC	23.5%	75	19%	16.04%	29.03%	29.2%	28.8%
NF	1.6%	7	2%	2.25%	4.08%	4.1%	4%
NB	2.3%	10	2.8%	3.12%	5.64%	5.7%	5.6%
NS	2.9%	11	3.2%	3.50%	6.32%	6.4%	6.3%
PE	0.4%	4	1.2%	1.34%	2.42%	2.4%	2.4%
YT	0.1%	1	0.3%	.39%	.71%	0.7%	0.7%
NT	0.1%	1	0.3%	.39%	.71%	0.7%	0.7%
NU	0.1%	1	0.3%	.39%	.71%	0.7%	0.7%

[2]All results for parliaments are computed using the algorithms from [4]

Each province and territory is assigned a number of seats in parliament for which candidates are elected though popular vote elections in the ridings within each province. The measures of power we defined have been applied using the number of seats as weightings and the parliamentary quota of 155 out of 308 seats.

The first thing to note is the similarity between population distribution and the SSI and NBI. This seems to demonstrate that the distribution of power in terms of voting blocks among provinces is, in fact, approximately divided by population.

3.1.1 The 2004 Federal Election

In the 2004 federal election, Canada elected a minority government for the first time since 1979. In fact there have been only 10 previous minority governments, the most recent prior to 2004 ended in a vote of no confidence shortly after it began. The 2004 parliament lasted 2 years before finally being defeated in a vote of no confidence. However, the Liberals were successful in forming coalitions in order to pass their budget and other pieces of legislation.

The results and power indices for the 2004 election are:

Party	Seats	Pop Vote	SSI	NBI	ABI	PIA	PPA
Lib.	135	36.7%	45%	44%	68.75%	73.33%	64.7%
Cons.	99	29.6%	20%	20%	31.25%	33.33%	29.4%
BQ	54	12.4%	20%	20%	31.25%	33.33%	29.4%
NDP	19	15.7%	11.67%	12%	18.75%	20%	17.6%
Ind.	1	0.3%	3.33%	4%	6.25%	6.67%	5.9%
Other	0	5.3%	0	0	0	0	0

First we see that the BQ has nearly 3 times the seats of the NDP which has more of the popular vote. The BQ only ran in Quebec and won 54 of Quebec's 74 seats whereas the NDP ran in nearly every riding Canada and spreads it's vote more thinly. This demonstrates why overall popular vote, much like weighting, is not usually an effective measure of power for systems such as this.

We also see how a single seat holds a disproportional amount of power in this parliament. In this case the minimal coalition the Liberals can form is a coalition with the NDP and single independent. This is also an attractive option because it forms the smallest possible coalition, resulting in a minimal sharing of power as described in [3]. In fact, the independent member in question was actually the deciding factor during a crucial vote.

In this situation a change of a single seat can also dramatically affect the balance of power. During this government a member of the Conservative party crossed the floor to the Liberals, possibly to capitalize on the unique opportunity for leverage that single seat held in this parliament. As a result we can also say that the power of a single seat can be fragile when the situation can change dynamically in this way.

3.1.2 2006 Federal Election

The 2006 election resulted in a second minority government but this time for the Conservatives. Just as in 2004 an independent was elected and once again their vote had pivotal power.

Party	Seats	Pop. Vote	SSI	NBI	ABI	PIA	PPA
Cons.	124	36.3%	40%	38.5%	62.5%	62.5%	62.5%
Lib.	102	30.2%	23.33%	23.1%	37.5%	37.5%	37.5%
BQ	51	10.5%	23.33%	23.1%	37.5%	37.5%	37.5%
NDP	29	17.5%	6.67%	7.7%	12.5%	12.5%	12.5%
Ind.	1	0.5%	6.67%	7.7%	12.5%	12.5%	12.5%
Other	0	5%	0	0	0	0	0

We should explain that the Liberals actually won 103 seats but one of those is the speaker of the house who does not vote unless breaking a tie in which case he will vote with the government.

We see yet another example of how seats are not an accurate measure of power. An independent with a single seat can be pivotal in as many coalitions as the NDP with 29 seats, even more seats then they had in 2004.

Not surprisingly, immediately after the election and before a single session of parliament could be held a member of the Liberal party crossed the floor to join the Conservatives. The Liberals now had only 101 seats and the Conservatives 125 which results in the following.

Party	Seats	Pop. Vote	SSI	NBI	ABI	PIA	PPA
Cons.	125	36.3%	50%	50%	75%	75%	75%
Lib.	101	30.2%	16.67%	16.67%	25%	25%	25%
BQ	51	10.5%	16.67%	16.67%	25%	25%	25%
NDP	29	17.5%	16.67%	16.67%	25%	25%	25%
Ind.	1	0.5%	0	0	0	0	0
Other	0	5%	0	0	0	0	0

As in the previous government, the result is a significant change in power from one single MP. The new balance of power leaves the independent without any power to form coalitions and the NDP are even with the Liberals and BQ. Each party is capable of forming a winning coalition or blocking as a united group.

3.2 Canadian Constitutional Amendment Scheme

The Canadian constitutional amendment scheme proposed in 1971 is an example of a non-weighted voting scheme. The reason is that it requires multiple sets of provinces to support an amendment rather than a simple majority or quota. The proposed scheme required that Ontario and Quebec, at least 2 out of the 4 maritime provinces and either all 3 prairie provinces or simply B.C. and one prairie province to support an amendment. In this scheme Ontario and Quebec have a veto on any amendment, without their support the amendment won't pass. The actual amendment scheme used in Canada is much simpler requiring that a 2/3 majority of the provinces support it and that the supporting provinces contain at least 50% of the population. We can demonstrate the proposed scheme is not a weighted voting game by forming these two coalitions,

$$C_1 = \{BC, AB, ON, QC, NB, NF\},$$
$$C_2 = \{BC, MB, ON, QC, NB, PE\}.$$

Both coalitions would pass the amendment. Now let us switch AB and PE to the other coalition to form,

$$C_1^* = \{BC, ON, QC, NB, NF, PE\},$$
$$C_2^* = \{BC, AB, MB, ON, QC, NB\}.$$

Clearly neither of these new coalitions will pass the amendment, the first does not have the western majority and the second does not have the maritime majority.

A listing of all passing coalitions is given in [6] which computes the SSI for each province and compares this to their populations to give a value of 'power per person'. We now represent those results and include results for the NBI as well as the NBI for the actual current amendment scheme[3].

Province	SSI('71)	NBI('71)	NBI(actual)	Pop. '72	Pop '06
B.C.	12.50%	16.34%	10.7%	9.38%	13.2%
AB	4.17%	5.45%	10.2%	7.33%	10.3%
SK	4.17%	5.45%	9.1%	4.79%	3.1%
MB	4.17%	5.45%	9.1%	4.82%	3.6%
ON	31.55%	21.78%	14.2%	34.85%	38.9%
QC	31.55%	21.78%	10.7%	28.94%	23.5%
NF	2.98%	5.9%	9.1%	2.47%	1.6%
NB	2.98%	5.9%	9.1%	3.09%	2.3%
NS	2.98%	5.9%	9.1%	3.79%	2.9%
PE	2.98%	5.9%	8.8%	0.54%	0.4%

The power in this proposal favors Ontario and Quebec more than the actual scheme. The actual scheme has a relatively even distribution of power among the provinces.

[3]Nunavut is omitted here to be consistent with the results from [6]

4 Conclusion

This work has defined the more common measures of power for voting games and demonstrated them using real examples. The power indices used give values indicative of an agents power to involve themselves in forming coalitions which can be an effective tool to understand real power in the parliamentary system.

The results from this work compare the different indices using real examples to illustrate the nature of measuring power in this way. We see how the power indices measure the ability of agents to achieve their desired result which depends as much on how seats are divided among other agents as it does on how many seats a particular agent gets. Furthermore we see how disconnected real power can be from the seats won by a party or even more the popular vote of a party.

Power indices provide a more refined way of looking at power than simply considering indicators such as seats in parliament or popular vote. While it does not capture the socio-political aspects of political coalitions it does provide an effective measure of potential power which can be considered like a form of leverage which is useful when considering how parties may negotiate or compromise to form these coalitions.

References

[1] J. Banzhaf. Weighted voting doesn't work: A mathematical analysis. *Rutgers Law Review*, (19):317–343, 1965.

[2] James S. Coleman. *Control of Collectivities and the Power of a Collectivity to Act.* Gordon and Breach, New York, 1971.

[3] Jr., Philip D. Straffin and Grofman, Bernard. Parliamentary coalitions: A tour of models. *Mathematics Magazine*, 57(5):259–274, nov 1984.

[4] Dennis Leech and Robert Leech. Computer algorithms for voting games. *Online: http://www.warwick.ac.uk/ ecaae/index.html*, December 2006.

[5] J. Malkevitch. Voting games: Part ii. *AMS: Feature Column: http://www.ams.org/featurecolumn/archive/voting.games.two.html*, October 2004.

[6] D. R. Miller. A shapley value analysis of the proposed canadian constitutional amendment scheme. *Canadian Journal of Political Science / Revue canadienne de science politique*, 6(1):140–143, mar 1973.

[7] L.S. Penrose. The elementary statistics of majority voting. *Journal of the Royal Statistical Society*, (109):53–57, 1946.

[8] Shapley, L.S. and Martin Shubik. A method for evaluating the distribution of power in a committee system. *American Political Science Review*, (48):787–792, 1954.