The difficulty of fairly allocating divisible goods

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December 27, 2006

1 Introduction

It is often necessary to divide a resource among a number of agents, and to do so in a way that does not leave any of the agents feeling cheated. There are a number of published methods for dealing with the problem of fairly dividing goods. Many of these methods are very successful, capable of finding allocations that satisfy the participating agents and distribute the goods being disputed in a way that they can agree to. These methods have done well in many real-world situations–from splitting desserts between squabbling siblings, to facilitating divorce settlements, to deciding international fishing rights.

There are situations in which goods cannot be fairly *divided*, but must be allocated to agents in a way that they consider fair. Unfortunately, common methods of fair division cannot be easily applied, if at all, in cases where the items are of lesser or no value when taken apart. The applicability and likely success of an allocation method depends on whether or not the items to be allocated are divisible or indivisible. Finding agreeable rules for an envy-free division of a set of indivisible items is difficult. Many of the methods currently published do not meet the same criteria that divisible-goods methods do and are susceptible to strategic manipulation. Although there are ways of avoiding the most difficult situations involving indivisible goods, the methods remain fragile and apply in a narrower range of cases than do methods where items can be divided.

2 Definitions

2.1 Divisibility

Divisible goods can be divided into smaller parts or portions and retain value. These are goods such as cakes or fishing rights. Divisible goods need not be homogeneous: imagine a marble cake, where each slice contains a different amount of chocolate or vanilla batter, or areas of the ocean where populations of fish are more or less likely to be. These heterogeneous goods are also divisible, although the allocation is made more somewhat more difficult: some agents may prefer pieces with more vanilla than chocolate.

Indivisible goods are things that have no value when they are broken into smaller items, or simply cannot be divided. For example, a single compact disc cannot be divided between two agents; the disc becomes unplayable when it is broken in half.

2.2 Fairness

A method can be said to be fair when the allocation it produces meets the following criteria.

Efficient (or **Pareto optimal**) allocations leave little to nothing remaining of the goods once they have been divided; there no way of improving the outcome for an agent without worsening the outcome for at least one other agent.

Envy-free allocations result in each agent being at least as happy with its share of the goods as it would be with any of the other agents' shares.

Equitable allocations leave all agents feeling as if they have received a fair share. For example, a method that would leave two siblings feeling as if they had received 60% of a cake would be equitable. If one sibling felt it had been favoured somehow and had received 80% of the cake, while the other believed it had received 60%, the allocation would not be equitable. Equitability can difficult to assess in many cases, since it is quite subjective.

Although not necessary to achieve a fair outcome, robustness is another desirable property in an allocation method. **Robust** methods cannot be easily worked-around or fooled in a way that would produce a better outcome for an agent. These are methods that, by design, promote agents to behave in a way that produces an allocation that meets the criteria above.

3 Methods for divisible goods

3.1 Cake-cutting methods

Cake-cutting methods are so called because they are often demonstrated using the example of a cake. The simplest and most widely known is the **divide-and-choose** method, where one of two agents divides the cake into two pieces and the second agent chooses which piece they will take. What remains is for the first agent. The divide-and-choose method is quite robust. Not creating two fair partitions is likely to harm the dividing agent; the distribution of tasks pressures the agent responsible for the division to create two equal partitions. Unfortunately, the divide-and-choose method does not work when there are more than two agents.

There are cake-cutting methods that can be applied in situations involving more than two agents, many of which also spread the tasks of division and selection across agents to promote fair behaviour and outcomes. A family of these are known as **moving knife** methods. These involve knives that move from left to right across a cake until a certain agent calls "stop." The point at which a knife is told to stop is marked in some way, and the procedure is repeated with another agent being given the responsibility to call the stop. At times, agents may be asked which partition (the boundaries of which are the marks) they prefer. The number of calls, cuts, and knives necessary, as well as the order in which agents are allowed to call or are asked their preferences, change according to the number of agents involved. For example, when dividing a cake among two agents, it is necessary to perform only one cut, one call, and to ask only one agent's preference. With three agents, it may be possible to divide the cake with two cuts, calls, and questions, but may require four. Procedures for four or more agents may require up to eleven.

Cake-cutting methods are generally efficient (all of the cake is divided among the agents, so there is no way for one agent to have more cake without taking it from another agent) and envy-free (every agent is satisfied with its piece), and, although difficult to ascertain in all cases, equitable (agents believe they have received a fair portion of cake). There are a two assumptions that must be made in order for these properties to hold. Firstly, the good being divided must be able to be divided at any point, that the value of a piece is continuous and proportional only to its size. Secondly, that agents will not make any mistakes in judgement when deciding among pieces.

3.2 Adjusted winner

It is often necessary to divide not a single, continuous good, but several goods among agents. Consider two siblings, a brother and sister, the only surviving kin of a rich old uncle who has passed away. The uncle has left behind a car, house, antique clock, and a number of investments, but not a will. Using the **adjusted winner** method, the brother and sister are list of all of the uncle's possessions to be divided. Each is given 100 points to assign to the items, giving more points to the items they prefer most. Their point assignments are given in Table 1.

Item	Brother	Sister
Car	25	15
House	30	47
Clock	30	20
Investments	15	18

Table 1: Points assigned for each of the uncle's items

Their lists of valuations are compared, and where one of their valuations is greater than the other's for an item, they are awarded that item. The brother would win the car and clock, 55 of his points, and the sister would win the house and investments, 65 of her points. This is then adjusted by dividing the item that they both valued most similarly, the investments, so that they both end up with the portions worth the same number of points. That is, the brother is given 30.3% of the investments, and both siblings receive roughly 60 points worth of the items.

The adjusted winner method is efficient and envy-free. The method is also equitable, since it is based on finding an allocation that leaves both agents with the same amount of value. The adjusted winner method is reasonably robust. When one agent knows how the other agent will assign its points, it is possible for that agent to report its points dishonestly and improve its own portion unfairly. This weakness, however, is quite small, since it is more likely to damage the dishonest player when it does not know the other agents' *exact* assignment of points. The method requires another two assumptions in order to be fair. Firstly, the disputed goods must be discrete and independent: the agents must value each good without considering them in combination with any other. (For example, agents dividing a collection of CDs will value each disc independently, including ones that may be a part of a set. An agent would value having only the first of the two discs in Little Richard's *Greatest Hits* as much as it would value that disc if it had both.) Secondly, the method assumes that agents report assign their points honestly.

4 The trouble with indivisible goods

The methods described above are fair and, excepting the possibility of agents making mistakes of judgement or being unreally aware of others values for items, quite robust. The cake-cutting methods have been used in a variety of real-world situations, including many international agreements regarding the rights of companies to fish or mine the ocean floor [2, 4]. The adjusted winner procedure is often used in business negotiations [6, 2]. Beyond being efficient and equitable, adjusted winner also reduces the amount of division necessary to resolve allocations–only the most closely valued item is compromised.

Sadly, the applicability and benefits are lost when the goods to be allocated cannot be divided. A simple example, involving two agents and a single good, is the tale of King Solomon and the two mothers. Two women approach Solomon both claiming to be the mother of a young child. Solomon is asked to decide which of the two is the child's true mother. Solomon calls for a sword, telling the women that he is going to split the child in two so that each of the women will have half. On hearing this, one of the women withdraws her claim, not wanting the child to be harmed. Solomon concludes she must be the child's mother and grants her custody of the child.

Although Solomon's solution is effective and reasonable, it illustrates one of the trade-offs necessary when allocating indivisible goods: by giving the child to one of the two agents, Solomon's method is efficient, but certainly not envy-free. Were he to keep the child himself, the result would be envy-free, but inefficient. Indeed this trade-off between efficiency and envy-freeness and the inability of the adjusted winner method's way of assigning values to items are two well known problems with indivisible-goods methods [6].

4.1 Adapting divisible-goods methods

Cake-cutting methods can be adapted to indivisible goods, but cannot guarantee results as effective or appealing as they do in divisible goods. In the example of the siblings dividing their dead uncle's goods (see Section 3.2), where the investments cannot be divided and all other items cannot be sold, the goods could be listed from left to right, with a sort of moving marker–the "knife"–moving over them. One of the siblings, say, the brother, calls "stop" when the marker divides the goods, in his eyes, evenly. The sister did not call, and so does not consider the items on one side to be worth as much as the items on the other. The brother must then swap items from one side of the marker to the other such that, to him, they remain reasonably the same value. When this swapping results in a acceptably even division, the sister tells him so. Each sibling is then randomly given a side. This method of division can result in reasonable allocations, but not guarantee equitable or envy-free ones, particularly in cases where there are a small number of goods, or where many of them are similarly valued. (The siblings in the example will always be at least 10 points apart in the end.)

The adjusted winner method can also be used, although without the ability to divide contentious items into smaller pieces, the final allocations will have to decide whether the disparity in value between agents will be greater if an agent is given that item (meaning the allocation will not be envy-free), or if that item is simply not given to any agent (meaning the allocation will not be efficient).

5 Improving indivisible-goods methods

5.1 Payouts

Perhaps the simplest way to avoid the difficulties with indivisible goods is to alter the situation to include a divisible good, such as money. Efficient, but not envy-free methods can compensate agents with smaller allocations by paying them (mechanism-to-agent payouts), or having agents with enviable allocations pay them (agent-to-agent payouts).

Alkan, Demange, and Gale [1], found that for any amount of money large enough, there exist allocations of indivisible goods among agents that are efficient and envy-free. This is true regardless of the number of agents or goods, so long as they are both greater than zero. When there are at least as many agents as there are goods, then the agents will be strictly better off as the amount of money increases.

5.2 Redefining fairness

Payouts and the mechanisms necessary for calculating them may be unappealing or unfeasible in certain circumstances. Herreiner and Puppe suggest a procedure that aims to maximize equitability and efficiency without compensating for lacking goods with money. Their *descending demand procedure*works under a different definition of equity, aiming to maximize the good for the worst-off agent (the *maxmin* division [5]). Each agent, submits an preference ordering on the set of goods to be divided. The procedure then iterates breadth-wise through each of these orderings, checking to see if it is possible to assign all the agents their most preferred item, or their second most preferred, etc. The procedure stops (in iteration k) when it comes across an agent's ordering that allows it to assign

- That agent its kth most preferred item,
- All agents before their kth most preferred item or better,

- All agents after their k-1'th most preferred item or better.

The details of the procedure for finding such allocations that are efficient is given in [3]. Unfortunately, it is not possible to alter the procedure to guarantee envy-free allocations will be found (indeed, the authors suggest that the procedure is likely to miss these allocations). What is interesting is how the method is capable of finding efficient allocations in which the disparity between agent's allocations is reduced in a certain sense: the distance between the greatest and least of the final allocations is minimized, reducing the intensity of the potential envy while guaranteeing no agent could be made better off.

Brams, Edelman, and Fishburn [5] carry this further, testing the fairness of methods involving not only the maxmin form of equity as in the descending demand procedure, but also maximizing the aggregate of all agents values for their allocations (*maxsum*), or a form of iterative maxmin called *equimax*, where the procedure attempts to maximize the value of the agent with the lowest valued allocation, and then to maximize for the next-lowest, given the first maximization, and so on. These altered definitions of fairness cannot guarantee efficient or envy-free allocations (indeed, maxsum can lead to very inefficient and envy-inducing allocations), but can be useful when dealing with indivisible-goods in contexts where the final values of agents' allocations are most important.

6 Conclusion

There are a number of generally applicable, strategy-proof methods for dividing goods among agents in a way that distributes all of the goods that there are to distribute, that does not leave them feeling envious of one another, and that leaves them with the sensation of having been treated equitably. These methods are primarily applicable to goods that can be divided arbitrarily, sliced or shared in some way. When the goods are indivisible, these methods do not apply in the same way: they cannot deal with goods efficiently, or can result in envious agents. It becomes necessary to include some other way of compensating for the differences in agents' allocations in these conditions, either by including some form of payoff, or by altering the goals of the method of division.

References

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