A Survey on Supermodular Games

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Abstract

Supermodular games are an interesting class of games that exhibits strategic complementarity. There are several compelling reasons like existence of pure strategy nash equilibrium, dominance solvability, identical bounds on joint strategy space etc. that make them a strong candidate for game theoretic modeling of economics. Supermodular games give a sound framework for analysis as it is based on a rich mathematical foundation of lattice theory and comparative statics. Supermodular games are also important for mechanism designers as it is observed that supermodularity has a strong connection with convergence to the nash equilibrium.

1 Introduction

Supermodular games are characterized by strategic complementarity [1]. The strategy space of every player is partially ordered and the marginal utility in playing a *higher* strategy increases when the opponents also play *higher* strategy. If the strategy space of a player is multidimensional then the marginal utility in increasing one dimension of the strategy also increases with the increase in other dimensions.

Strategic complementarity is a common phenomenon in many real life situations. For example, an R&D race: if company A's main rival company B starts spending more money on research (takes a higher strategy) it becomes more advisable for company A to spend more money on research. Or a bank run: if more people are withdrawing their money from the bank where you have kept your savings it becomes more profitable for you to do the same.

The formal definition of supermodular game will require a little digression to a term 'increasing difference'.

A function f : X $\times$ T $\rightarrow \Re$ has increasing differences in (x, t) if $\forall$ $x' \geq x$ and $t' \geq t$, $f(x', t')$ - $f(x, t')$ $\geq$ $f(x', t)$ - $f(x, t)$.

It implies that if $f(x, t)$ has increasing differences in (x, t) the marginal benefit for taking a higher x (i.e. x' instead of x) increases when t (i.e. t' instead of t) increases.

A game [1] (N, S { u_i : i $\in$ N}) where,

- N is a finite set of n players;
- S is a set of feasible joint strategies;
- u_i: S $\times$ N $\rightarrow \Re$ is a real valued function mapping utility for player i when joint strategy s $\in$ S is played.

is a supermodular game when,

- S is a sublattice of $\Re^m$ (see appendix)
- payoff function u_i has increasing differences in (s_i,s_{-i}).

A three player game, with the following utility function [8] is an example of a supermodular game. Each player has three ordered strategies 1,2,3. For every outcome, all the players have equal utility as indicated by the utility function.

$$u(i_1, i_2, i_3) = \overset{i_1 = 1}{\begin{pmatrix} 100 & 79 & 58 \\ 85 & 67 & 49 \\ 25 & 19 & 13 \end{pmatrix}} \quad \overset{i_2 = 2}{\begin{pmatrix} 65 & 51 & 37 \\ 55 & 43 & 31 \\ 15 & 11 & 7 \end{pmatrix}} \quad \overset{i_3 = 3}{\begin{pmatrix} 30 & 23 & 16 \\ 25 & 19 & 13 \\ 5 & 3 & 1 \end{pmatrix}}$$

The utility function has increasing differences in opponents strategy as u_1(3,2,1) - u_1(1,2,1) = 19 - 79 = -60 $\geq$ u_1(3,1,1) - u_1(1,1,1) = 25 - 100 = -75.

The name supermodular comes from the fact that the utility function is always supermodular (see appendix) satisfying the condition, $f(z \vee z') + f(z \wedge z') \geq f(z) + f(z')$ where $z \vee z'$ and $z \wedge z'$ denotes the component-wise maximum (join) and the component-wise minimum (meet) of z and z' respectively.

	B	F
B	2, 1	0, 0
F	0, 0	1, 2

Figure 1: Battle of the Sexes game

Several well known games like prisoner's dilemma, the Battle of the Sexes can be modeled as supermodular games by putting an ordering on the strategy space. The Battle of the Sexes game can be treated as a supermodular game [5] when both the players consider F being a higher strategy to B. The way the strategies are ordered often determine whether a game is supermodular or not. If we take for player 1 (row player) the strategy ordering is F > B and for player 2 (column player) it is B > F, the game is no more supermodular. The utility function is violating u_1(B,F) + u_1(F,B)$\geq$ u_1(B,B) + u_1(F,F) that is a necessary condition for being supermodular.

Section 2 will present some economic situations that can be modeled as a supermodular game. In Section 3 various interesting results regarding the solution concepts of the game are discussed.

2 Examples of Supermodular Games

2.1 Arms Race Game

Consider a set of countries (players) N = 1,2,3,...,n engaged in an arms race [1]. Each country i selects its level of arms x_i and the utility function u_i has increasing differences in (s_i,s_{-i}). This actually means that the perceived value of additional arms to any country i increases with the arms level of other countries; i.e., additional arms are considered to be more valuable when the military capability of one's adversaries are greater.

2.2 Investment and New Technology Adoption

Investment game is another suitable example. In this game, there are N firms I_1, I_2,...,I_n making investment $s_i \ \epsilon \ $ 0,1 and the payoffs are,
$u_i(s_i,s_{-i}) = \pi(\sum_{j=1}^{j=n} s_j)$ when $s_i \neq 0$
$= 0$ when $s_i = 0$
where π is increasing in aggregate investment.

Similarly, new technology adoption can also be viewed as a supermodular game. It becomes more profitable for a particular company to adopt a new technical standard when other companies are also doing the same.

2.3 Pretrial Negotiation

In this setup [4], before going for the trial, the defendant and plaintiff sit together for a negotiation. The entire process can be viewed as a simple flowchart presented in fig. 2.

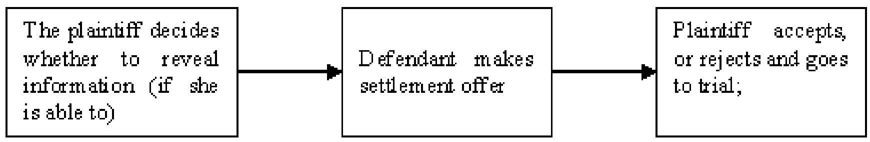

Figure 2: Pretrial negotiation

The strategy of the plaintiff is denoted by the highest level of expected damage for which she remains silent. The strategy of the defendant is denoted by the amount she offers to a silent plaintiff. Clearly, it is more profitable for the plaintiff to adopt a high strategy when the settlement offer to her is high. It is also more profitable for the defendant to adopt a high strategy when the plaintiff adopts a high strategy. This is because if the plaintiff rejects any offer that means the plaintiff believes she has enough private information to get a better deal from the trial. Hence, this bayesian game sets up a nice example of supermodular games.

3 Solution Concepts and their properties

3.1 Existence of Nash Equilibrium

Tarski's fixed point theorem [9] proves the existence of Pure Strategy Nash Equilibrium(PSNE) in supermodular games. If there are two or more PSNE, then the set of Nash Equilibria will be a complete lattice. In case of a two player game, with a total ordering of the strategy space, the set of PSNE will be a sublattice (see appendix) [6]. Since the sublattice is closed under the 'join' and 'meet' operations other existing PSNEs in a game can be identified once two of them are computed.

For economic modeling, existence of pure strategy nash equilibrium is a desirable property as it is intuitively compelling for many of the economic situations to have PSNE. Moreover, when multiple equilibria exist one equilibria may be the more desired one. Conditions to enforce the desired equilibrium needs to be analyzed. For example, there exists two PSNEs in the arms race game [1]-an 'arms race' equilibrium in which all the countries acquire new weapons, and a 'detente' equilibrium in which none of them acquires any. Clearly, in this case, the 'detente' equilibrium is the more desired one. Modeling arm's race as a supermodular game helps in understanding the conditions when 'detente' equilibrium can be enforced.

3.2 Identical Bounds on Joint Strategy Space

Milgrom and Roberts [2] presents an interesting result related to the upper and lower bound of serially undominated strategy found in a supermodular game. If $\bar{a}$ and $\underline{a}$ are respectively the highest and lowest PSNE of a supermodular game and U denotes the set of strategies that survive iterated deletion of strictly dominated strategies (ISD) then, sup U = $\bar{a}$ and inf U = $\underline{a}$. Hence, for a supermodular game, Nash Equilibria and ISD both give identical bounds on the joint strategy space. Moreover, if MSNE denotes the set of all Mixed Strategy Nash Equilibria and CE denotes the set of all Correlated Equilibria then, Supp MSNE $\subset$ U and Supp CE $\subset$ U. Thus all the solution concepts provide identical bounds on the joint strategy space.

Analyzing non cooperative games with a single general solution concept is not a feasible idea as the game situation varies in different games[2]. Games that are played between complete strangers with no precedents players will find it hard to gauge how rational their opponents are and hence selecting a strategy depending on the rule of the game and general knowledge of human behavior will be difficult. In this situation, correlated equilibria provides a better alternative to Nash Equilibria. Games that have long precedents and are played between players who know each other will show different behavior and converging to a Nash Equilibrium using adaptive learning mechanism seems feasible. There can also be situations that don't belong to any of these two extremes instead fall somewhere in between these two classes. Supermodular game is of particular interest because all the solution concepts provide identical bounds on the joint strategy space.

3.3 Dominance Solvability

The concept of providing an upper bound and lower bound is often helpful in finding the *interesting* region in the joint strategy space of the game. In some cases it really narrows down the joint strategy space. But often the predicted range is very wide [2] and sometimes same as the upper and lower bound of the joint strategy space if these two bounds are Nash Equilibria. In those cases, this property is not of much help. An interesting result is observed when a supermodular game has only one unique PSNE. In this case, the $\bar{a}$ and $\underline{a}$ will coincide and the infimum and supremum for U (the set of strategies that survive iterated deletion of strictly dominated strategy) will become identical indicating that U will be a singleton. This implies that only the unique PSNE will survive iterated removal of strictly dominated strategy. Since MSNE and CE are both subsets of U all the solution concepts will give identical solution. This property is called dominance solvability.

In game theory several refinements of Nash Equilibrium exist. The various iterative solution concepts also have their own versions of *rationalizable* strategy. If a game is dominance solvable then all these alternatives produce the same unique result.Thus, dominance solvability is a desirable property for a game as almost all the game theorists *agree* about their predictions of the behavior of the players.

3.4 Instability of Mixed Strategy Nash Equilibrium

The MSNE in supermodular games is generally unstable [5]. Consider, the example of the Battle of Sexes game as given in the introduction. The game has two pure strategy NE (B,B) and (F,F). The only mixed strategy NE in this game is ($\frac{2}{3}$B+$\frac{1}{3}$F, $\frac{1}{3}$B+$\frac{2}{3}$F).

For stability of the mixed strategy NE, it is crucial for player 1 to believe that player 2 is exactly playing $\frac{1}{3}$B+$\frac{2}{3}$F. If player 1 believes player 2 is playing F with probability slightly greater than $\frac{2}{3}$ (say $\frac{2}{3} + \varepsilon$) then player 1 will strictly prefer F. Similarly, if player 2 believes player 1 will play F with probability $\frac{1}{3}$ $+ \eta$ then player 2 will also strictly prefer F. Even if we take ε and η to be very close to zero, in a repeated play, after each iteration one players belief about opponents strategy will get revised and ε and η will keep on getting more weightage on subsequent iterations. In this way, gradually the game will shift to a pure strategy NE (F,F). This example proves that a slight perturbation in the beliefs of the players can shift them from playing a mixed strategy NE to a pure strategy NE.

Analyzing MSNEs in a complex economic environment is not well understood [5] . The main contribution of this paper is if a problem can be modeled as a supermodular game, we don't need to worry about MSNEs any more.

3.5 Role of Supermodularity in Learning to Play the Equilibrium

Milgrom [2] and Vives [3] have observed that supermodular games are robust to many learning dynamics. The fact that all solution concepts yield identical bounds on the joint strategy space can be a reason for that. But how fast learning converges to an equilibrium is practically important.

Chen and Gazzale [7] presents a study that tries to identify the role of supermodularity in learning to play the equilibrium. In this study, a parametrized game is taken and the parameter is used to vary the degree of supermodularity of the game. It is observed that learning converges to the Nash Equilibrium when the game is *close* to the threshold of supermodularity. Games that are higher than the threshold of supermodularity don't converge significantly faster than games close to the threshold. But the convergence performance of games far below the threshold of supermodularity is significantly poor as compared to games close to the threshold. Hence, mechanism designers need not attempt for a high level of supermodularity to achieve fast convergence; anything close to the threshold is good enough.

4 Conclusion

Supermodular games are interesting for the following reasons. Firstly, it encompasses wide classes of games and myriad economic situations that involve complementarity. It gives us a framework to analyze them and identify the critical properties of the payoffs and action spaces. Existence of a pure strategy nash equilibrium is most often a desired concept in economic models and hence supermodular games increase the scope of game-theoretic modeling in economics. Secondly, since supermodular games are based on a strong mathematical foundation of lattice theory and comparative statics it simplifies the analysis, identifies regularity conditions necessary for obtaining a desired outcome. Thirdly, multiple equilibria situations [10] can be analyzed by ranking equilibria. Several interesting questions can be answered e.g. the direction an equilibrium shifts when a new player enters the game, or what can be the effect on a bank run if the solvency ratio is increased. Finally, supermodular games are important to mechanism designers as it is observed if a game is supermodular or close to the threshold of being so, the convergence rate to equilibrium is significantly high.

5 References

1)Topkis, D. (1998), "Supermodularity and Complementarity". Princeton University Press. Princeton

2) Milgrom, P. and Roberts, J. (1990), "Rationalizability and Learning in Games with Strategic Complementarities," Econometrica, 58, 1255-1278.

3) Vives, X. (1990), "Nash Equilibrium with Strategic Complementarities,"J. Math. Econ, 19, 305-321.

4) Shavell, S. (1989), "Sharing of Information Prior to Settlement or Litigation,"Rand Journal of Economics 20: 183-195.

5) Echenique, F. and Edlin, A. (2003), "Mixed Strategy Equilibria in Games of Strategic Complements are Unstable," Caltech Working Paper.

6) Echenique, F. (2003), "The equilibrium set of two-players games with complementarities is a sublattice", Economic Theory 22, 903-905.

7) Chen, Y. and Gazzale., R. (2004) "When Does Learning in Games Generate Convergence to Nash Equilibria? The Role of Supermodularity in an Experimental Setting".. American Economic Review vol. 94, No. 5, December 2004: 1505-1535

8) Sherstyuk, K. (1999), "Multisided Matching Games with Complementarities," International Journal of Game Theory, 28, 489-509.

9) Tarski, A. (1955), "A lattice-theoretic fixed point theorem and its applications". Pacific Journal of Mathematics 5:285-309.

10) Vives X. (2004), Complementarities and Games: New Developments," CEPR Discussion Paper 4742.

6 Appendix

Supermodular function: A function $f : \Re^k \rightarrow \Re$ is supermodular if for all z, z', $f(z \vee z') + f(z \wedge z') \geq f(z) + f(z')$ where $z \vee z'$ and $z \wedge z'$ denotes the component-wise maximum (join) and the component-wise minimum (meet) of z and z' respectively.

The three player utility function given in the introduction is supermodular as,
u_1(1,3,2) + u_1(2,1,2) = 37 + 55 = 92 $\leq$ u_1(2,3,2) + u_1(1,1,2) = 31 + 65 = 96.

Poset or partially ordered set: A partial order is a binary relation R over a set P which is reflexive, antisymmetric, and transitive, i.e., for all a, b, and c in P, we have that:

- aRa (reflexivity);
- if aRb and bRa then a = b (antisymmetry); and
- if aRb and bRc then aRc (transitivity).

A set with a partial order is called a partially ordered set. For a total order, antisymmetry and transitivity hold, but instead of reflexivity a new condition totality that is a ≤ b or b ≤ a is present.

Example: The set of natural numbers equipped with the lesser than or equal to relation. Its reflexive because a ≤ a. Antisymmetric because, if a ≤ b and b ≤ a then a = b. And its transitive as a ≤ b and b ≤ a implies a ≤ c.

Lattice: A lattice is a partially ordered set (or poset) whose nonempty finite subsets all have a supremum (called join) and an infimum (called meet).

Example: For any set A, the collection of all subsets of A (called the power set of A) can be ordered via subset inclusion to obtain a lattice bounded by A itself and the null set. Set intersection and union interpret meet and join, respectively.

Sublattice: A sublattice of a lattice L is a nonempty subset of L which is a lattice with the same meet and join operations as L. That is, if L is a lattice and M$\neq \emptyset$ is a subset of L such that for every pair of elements a, b in M both a$\wedge$b and a$\vee$b are in M, then M is a sublattice of L

Example: Under the usual ordering on $\Re^2$, the set T = {(0,0),(1,0),(0,1),(2,2)} is a lattice but not a sublattice of $\Re^2$ as because (1,0) $\vee$ (0,1) = (1,1) $\notin$ T.

Complete sublattice: A sublattice M of a complete lattice L is called a complete sublattice of L if for every subset A of M the elements $\wedge$ A and $\vee$ A, as defined in L, are actually in M.