Extensive Form Games

CPSC 532A Lecture 9

October 10, 2006

Lecture Overview

Recap

Perfect-Information Extensive-Form Games

Subgame Perfection

Formal definition

- (Ω, π) is a finite probability space
- for every agent i, divide Ω into a set of partitions $\mathbf{P_i} = \{P_{i,1}, \ldots, P_{i,k_i}\}$
  - For all i, $\bigcup_{j=1}^{k_i} P_{i,j} = \Omega$ and $j \neq j'$ implies that $P_{i,j} \cap P_{i,j'} = \emptyset$.
  - We'll use the partitions to indicate values of Ω that are indistinguishable for i.
- (Pure) strategy: $\sigma_i : \Omega \rightarrow A_i$
  - To capture our intuition about the partitions, we need the property that $(\omega, \omega' \in \mathbf{P}_i)$ implies that $\sigma_i(\omega) = \sigma_i(\omega')$

Definition

$(\Omega, \pi, \mathcal{P}, \sigma)$, is a correlated equilibrium when

$$\sum_{\omega \in \Omega} \pi(\omega) u_i \left(\sigma_i(\omega), \sigma_{-i}(\omega)\right) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i \left(\sigma'_i(\omega), \sigma_{-i}(\omega)\right).$$

Other remarks

- For every Nash equilibrium σ^* of a game $G = (N, A, u)$ there exists a correlated equilibrium $(\Omega, \pi, \mathcal{P}, \sigma)$ under which each agent $i \in N$ plays each action $a \in A_i$ with probability $\sigma_i^*(a)$.
- Not every correlated equilibrium is equivalent to a Nash equilibrium
  - thus, correlated equilibrium is a weaker notion than Nash
- Any convex combination of the payoffs achievable under correlated equilibria is itself realizable under a correlated equilibrium

Computing CE

$$\sum_{a \in A | a_i \in a} p(a) u_i(a) \geq \sum_{a \in A | a_i' \in a} p(a) u_i(a_i', a_{-i}) \quad \forall i \in N, \forall a_i, a_i' \in A_i$$

$$p(a) \geq 0 \quad \forall a \in A$$

$$\sum_{a \in A} p(a) = 1$$

- variables: $p(a)$; constants: $u_i(a)$

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$$\sum_{a \in A} p(a) = 1$$

- variables: $p(a)$; constants: $u_i(a)$
- we could find the social-welfare maximizing CE by adding an objective function

$$\text{maximize:} \quad \sum_{a \in A} p(a) \sum_{i \in N} u_i(a).$$

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Introduction

- The normal form game representation does not incorporate any notion of sequence, or time, of the actions of the players
- The extensive form is an alternative representation that makes the temporal structure explicit.
- Two variants:
  - perfect information extensive-form games
  - imperfect-information extensive-form games

Definition

A (finite) perfect-information game (in extensive form) is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u)$, where

- N is a set of n players
- $A = (A_1, \ldots, A_n)$ is a set of actions for each player
- H is a set of non-terminal choice nodes
- Z is a set of terminal nodes, disjoint from H
- $\chi : H \to 2^A$ is the action function
    - assigns to each choice node a set of possible actions
- $\rho : H \to N$ is the player function
    - assigns to each non-terminal node a player $i \in N$ who chooses an action at that node

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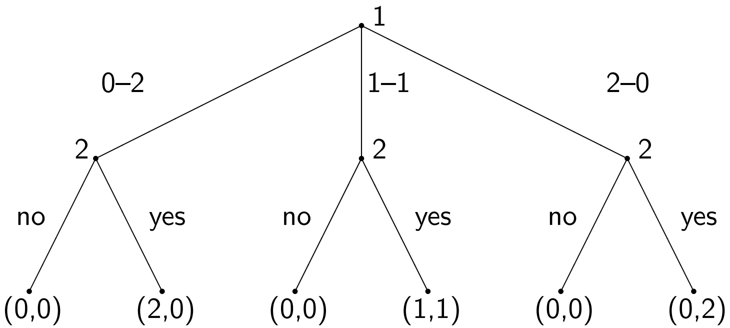
- $\rho : H \rightarrow N$ is the player function
  - assigns to each non-terminal node a player $i \in N$ who chooses an action at that node
- $\sigma : H \times A \rightarrow H \cup Z$ is the successor function
  - maps a choice node and an action to a new choice node or terminal node such that for all $h_1, h_2 \in H$ and $a_1, a_2 \in A$, if $\sigma(h_1, a_1) = \sigma(h_2, a_2)$ then $h_1 = h_2$ and $a_1 = a_2$
- $u = (u_1, \ldots, u_n)$, where $u_i : Z \rightarrow \mathbb{R}$ is a utility function for player i on the terminal nodes Z

Note: the choice nodes form a tree, so we can identify a node with its history.

Example: the sharing game



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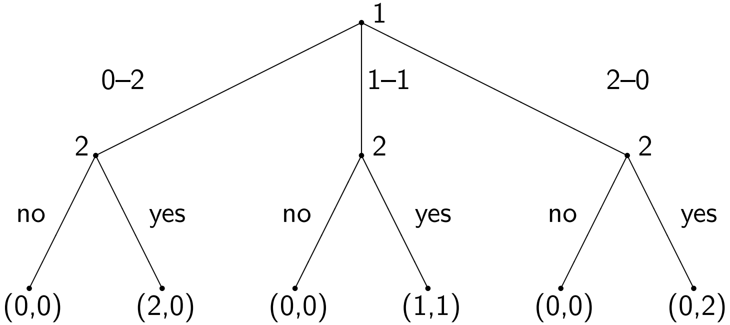

Play as a fun game, dividing 100 dollar coins. (Play each partner only once.)

Pure Strategies

- In the sharing game (splitting 2 coins) how many pure strategies does each player have?

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  - player 1: 3; player 2: 8

Pure Strategies

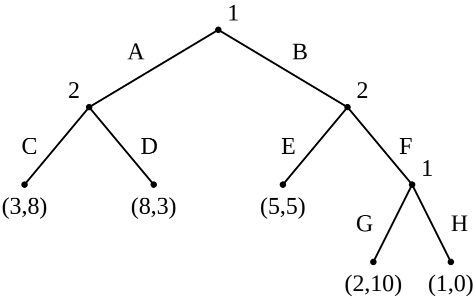
- In the sharing game (splitting 2 coins) how many pure strategies does each player have?
  - player 1: 3; player 2: 8
- Overall, a pure strategy for a player in a perfect-information game is a complete specification of which deterministic action to take at every node belonging to that player.

Definition

Let $G = (N, A, H, Z, \chi, \rho, \sigma, u)$ be a perfect-information extensive-form game. Then the pure strategies of player i consist of the cross product

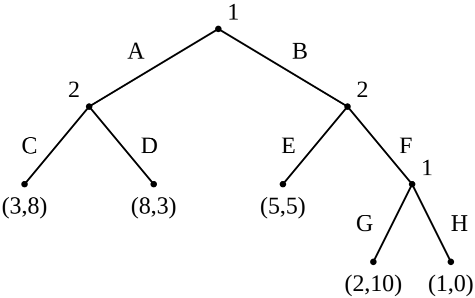
$$\bigtimes_{h \in H, \rho(h)=i} \chi(h)$$

Pure Strategies Example



What are the pure strategies for player 2?

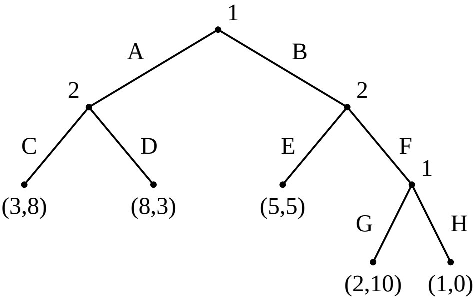
Pure Strategies Example



What are the pure strategies for player 2?

- $S_2 = \{(C, E); (C, F); (D, E); (D, F)\}$

Pure Strategies Example



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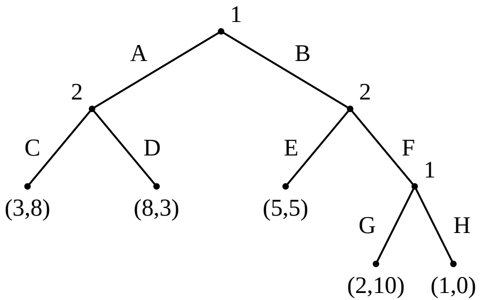
Pure Strategies Example



What are the pure strategies for player 2?

- $S_2 = \{(C, E); (C, F); (D, E); (D, F)\}$

What are the pure strategies for player 1?

- $S_1 = \{(B, G); (B, H), (A, G), (A, H)\}$
- This is true even though, conditional on taking A, the choice between G and H will never have to be made

Nash Equilibria

Given our new definition of pure strategy, we are able to reuse our old definitions of:

- mixed strategies
- best response
- Nash equilibrium

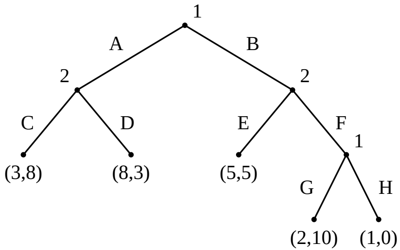
Theorem

Every perfect information game in extensive form has a PSNE

This is easy to see, since the players move sequentially.

Induced Normal Form

- In fact, the connection to the normal form is even tighter
  - we can "convert" an extensive-form game into normal form



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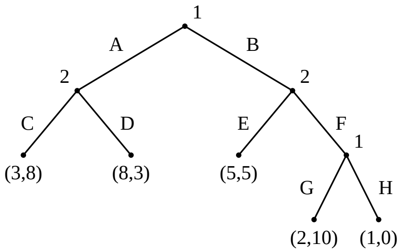

	CE	CF	DE	DF
AG	$3,8$	$3,8$	$8,3$	$8,3$
AH	$3,8$	$3,8$	$8,3$	$8,3$
BG	$5,5$	$2,10$	$5,5$	$2,10$
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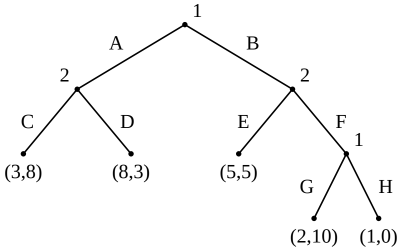

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- this illustrates the lack of compactness of the normal form
  - games aren't always this small
  - even here we write down 16 payoff pairs instead of 5

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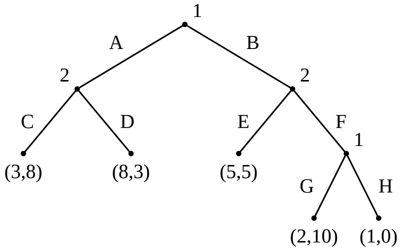

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- while we can write any extensive-form game as a NF, we can't do the reverse.
  - e.g., matching pennies cannot be written as a perfect-information extensive form game

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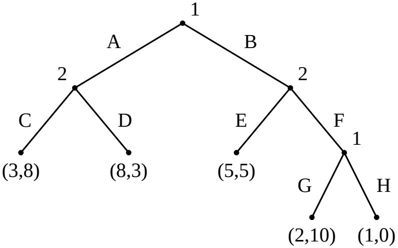

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- What are the (three) pure-strategy equilibria?

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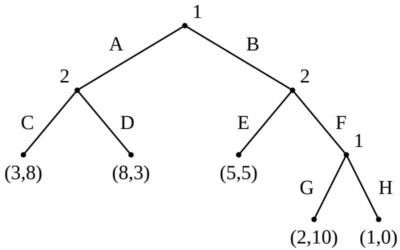

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- What are the (three) pure-strategy equilibria?
  - $(A,G),(C,F)$
  - $(A,H),(C,F)$
  - $(B,H),(C,E)$

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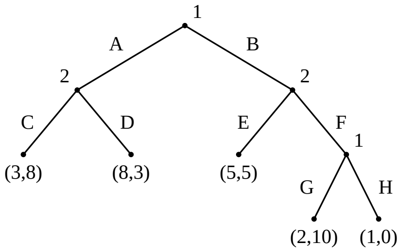

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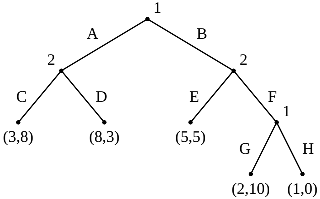
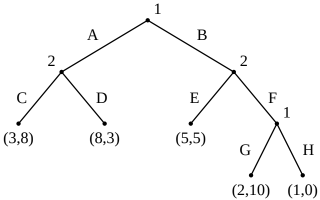


- There's something intuitively wrong with the equilibrium $(B, H), (C, E)$
  - Why would player 1 ever choose to play H if he got to the second choice node?
    - After all, G dominates H for him

Subgame Perfection



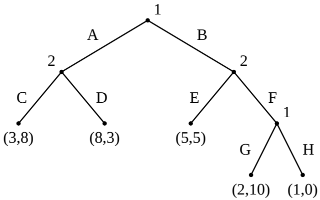
- There's something intuitively wrong with the equilibrium $(B, H), (C, E)$
  - Why would player 1 ever choose to play H if he got to the second choice node?
    - After all, G dominates H for him
  - He does it to threaten player 2, to prevent him from choosing F, and so gets 5
    - However, this seems like a non-credible threat
    - If player 1 reached his second decision node, would he really follow through and play H?

Formal Definition

- Define subgame of G rooted at h:
  - the restriction of G to the descendents of H.
- Define set of subgames of G:
  - subgames of G rooted at nodes in G

- s is a subgame perfect equilibrium of G iff for any subgame G' of G, the restriction of s to G' is a Nash equilibrium of G'
- Notes:
  - since G is its own subgame, every SPE is a NE.
  - this definition rules out "non-credible threats"

Back to the Example



- Which equilibria from the example are subgame perfect?

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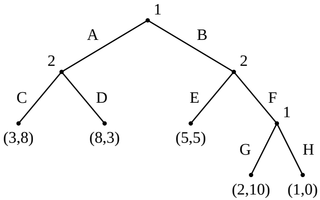

- Which equilibria from the example are subgame perfect?
  - $(A, G), (C, F)$ is subgame perfect
  - (B, H) is an non-credible threat, so $(B, H), (C, E)$ is not subgame perfect
  - (A, H) is also non-credible, even though H is "off-path"