Correlated Equilibria

CPSC 532A Lecture 8

October 5, 2006

Lecture Overview

Recap

Correlated Equilibrium

Computing Correlated Equilibria

Iterated Removal of Dominated Strategies

- This process preserves Nash equilibria.
  - strict dominance: all equilibria preserved.
  - weak or very weak dominance: at least one equilibrium preserved.
- Thus, it can be used as a preprocessing step before computing an equilibrium
  - Some games are solvable using this technique.
- What about the order of removal when there are multiple dominated strategies?
  - strict dominance: doesn't matter.
  - weak or very weak dominance: can affect which equilibria are preserved.

Is s_i strictly dominated by any pure strategy?

for all pure strategies $a_i \in A_i$ for player i where $a_i \neq s_i$ **do**
  $dom \leftarrow true$
  for all pure strategy profiles $a_{-i} \in A_{-i}$ for the players other than i
  do
    if $u_i(s_i, a_{-i}) \geq u_i(a_i, a_{-i})$ **then**
      $dom \leftarrow false$
      break
    end if
  end for
  if $dom = true$ **then return** $true$
end for
return $false$

- What the complexity of this procedure is $O(|A|)$.
- We don't have to check mixed strategies of the other players because of linearity of expectation

LP for determining whether s_i is strictly dominated by any mixed strategy

$$\begin{aligned} \text{minimize} \quad & \sum_{j \in A_i} p_j \\ \text{subject to} \quad & \sum_{j \in A_i} p_j u_i(a_j, a_{-i}) \geq u_i(s_i, a_{-i}) \qquad \forall a_{-i} \in A_{-i} \end{aligned}$$

- This is clearly an LP. Why is it a solution to our problem?
  - if a solution exists with $\sum_j p_j < 1$ then we can add $1 - \sum_j p_j$ to some p_k and we'll have a dominating mixed strategy (since utility was assumed to be positive everywhere)

Iterated elimination

Finding a single game where all strategies survive elimination of dominated strategies is polynomial-time. Other questions:

1. **(Strategy Elimination)** Does there exist some elimination path under which the strategy s_i is eliminated?
2. **(Reduction Identity)** Given action subsets $A'_i \subseteq A_i$ for each player i, does there exist a maximally reduced game where each player i has the actions A'_i?
3. **(Uniqueness)** Does every elimination path lead to the same reduced game?
4. **(Reduction Size)** Given constants k_i for each player i, does there exist a maximally reduced game where each player i has exactly k_i actions?

- For iterated strict dominance these problems are all in $\mathcal{P}$.
- For iterated weak or very weak dominance these problems are all $\mathcal{NP}$-complete.

Rationalizability

- Rather than ask what is irrational, ask what is a best response to some beliefs about the opponent
  - assumes opponent is rational
  - assumes opponent knows that you and the others are rational
  - ...
- Will there always exist a rationalizable strategy?
  - Yes, equilibrium strategies are always rationalizable.
- Furthermore, in two-player games, rationalizable $\Leftrightarrow$ survives iterated removal of strictly dominated strategies.

Lecture Overview

Pithy Quote

If there is intelligent life on other planets, in a majority of them, they would have discovered correlated equilibrium before Nash equilibrium.

– *Roger Myerson*

Examples

- Consider again Battle of the Sexes.
  - Intuitively, the best outcome seems a 50-50 split between (F, F) and (B, B).
  - But there's no way to achieve this, so either someone loses out (unfair) or both players often miscoordinate
- Another classic example: traffic game

	go	$wait$
go	$-100, -100$	$10, 0$
B	$0, 10$	$-10, -10$

Intuition

- What is the natural solution here?

Intuition

- What is the natural solution here?
  - A traffic light: a fair randomizing device that tells one of the agents to go and the other to wait.
- Benefits:
  - the negative payoff outcomes are completely avoided
  - fairness is achieved
  - the sum of social welfare exceeds that of any Nash equilibrium
- We could use the same idea to achieve the fair outcome in battle of the sexes.
- Our example presumed that everyone perfectly observes the random event; not required.
- More generally, some random variable with a commonly known distribution, and a private signal to each player about the outcome.
  - signal doesnt determine the outcome or others' signals; however, correlated

Formal definition

- (Ω, π) is a finite probability space
- for every agent i, divide Ω into a set of partitions $\mathbf{P_i} = \{P_{i,1}, \dots, P_{i,k_i}\}$
  - For all i, $\bigcup_{j=1}^{k_i} P_{i,j} = \Omega$ and $j \neq j'$ implies that $P_{i,j} \cap P_{i,j'} = \emptyset$.
  - We'll use the partitions to indicate values of Ω that are indistinguishable for i.
- (Pure) strategy: $\sigma_i : \Omega \to A_i$
  - To capture our intuition about the partitions, we need the property that $(\omega, \omega' \in \mathbf{P}_i)$ implies that $\sigma_i(\omega) = \sigma_i(\omega')$

Definition

$(\Omega, \pi, \mathcal{P}, \sigma)$, is a correlated equilibrium when

$$\sum_{\omega \in \Omega} \pi(\omega) u_i \left(\sigma_i(\omega), \sigma_{-i}(\omega)\right) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i \left(\sigma_i'(\omega), \sigma_{-i}(\omega)\right).$$

Existence

Theorem

For every Nash equilibrium σ^* of a game $G = (N, A, u)$ there exists a correlated equilibrium $(\Omega, \pi, \mathcal{P}, \sigma)$ under which each agent $i \in N$ plays each action $a \in A_i$ with probability $\sigma_i^*(a)$.

Proof. We show how to construct the correlated equilibrium from the given Nash equilibrium σ^*. Set Ω to be $A_1 \times \ldots \times A_n$, the joint action space of G. Set $\pi(a)$ to be $\prod_{i \in N} \sigma^*(a_i)$, the probability that joint action a will be played under the joint mixed strategy σ^*. Set $\mathcal{P}_{i,j}$ to be the set of joint actions in which player i takes action j. Then the correlated equilibrium strategy is $\sigma_i(\omega) = a$ for $\omega \in \mathcal{P}_{i,a}$. The fact that no agent can increase his utility by adopting some new strategy σ_i' follows directly from the fact that σ^* is a Nash equilibrium.

Other remarks

- Not every correlated equilibrium is equivalent to a Nash equilibrium
  - thus, correlated equilibrium is a weaker notion than Nash
- Any convex combination of the payoffs achievable under correlated equilibria is itself realizable under a correlated equilibrium
  - start with the Nash equilibria (each of which is a CE)
  - introduce a second randomizing device that selects which CE the agents will play
  - regardless of the probabilities, no agent has incentive to deviate
  - the probabilities can be adjusted to achieve any convex combination of the equilibrium payoffs
  - the randomizing devices can be combined

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Recap

Correlated Equilibrium

Computing Correlated Equilibria

Computing CE

$$\sum_{a \in A | a_i \in a} p(a) u_i(a) \geq \sum_{a \in A | a_i' \in a} p(a) u_i(a_i', a_{-i}) \quad \forall i \in N, \forall a_i, a_i' \in A_i$$

$$p(a) \geq 0 \quad \forall a \in A$$

$$\sum_{a \in A} p(a) = 1$$

- variables: $p(a)$; constants: $u_i(a)$

Computing CE

$$\sum_{a \in A | a_i \in a} p(a) u_i(a) \geq \sum_{a \in A | a'_i \in a} p(a) u_i(a'_i, a_{-i}) \quad \forall i \in N, \forall a_i, a'_i \in A_i$$

$$p(a) \geq 0 \quad \forall a \in A$$

$$\sum_{a \in A} p(a) = 1$$

- variables: $p(a)$; constants: $u_i(a)$
- we could find the social-welfare maximizing CE by adding an objective function

$$\text{maximize:} \quad \sum_{a \in A} p(a) \sum_{i \in N} u_i(a).$$

Why are CE easier to compute than NE?

$$\sum_{a \in A | a_i \in a} p(a) u_i(a) \geq \sum_{a \in A | a'_i \in a} p(a) u_i(a'_i, a_{-i}) \quad \forall i \in N, \forall a_i, a'_i \in A_i$$

$$p(a) \geq 0 \quad \forall a \in A$$

$$\sum_{a \in A} p(a) = 1$$

- intuitively, correlated equilibrium has only a single randomization over outcomes, whereas in NE this is constructed as a product of independent probabilities.
- To change this program so that it finds NE, the first constraint would be

$$\sum_{a \in A} u_i(a) \prod_{j \in N} p_j(a_j) \geq \sum_{a \in A} u_i(a'_i, a_{-i}) \prod_{j \in N \setminus \{i\}} p_j(a_j) \quad \forall i \in N, \forall a'_i \in A_i.$$

- This is a nonlinear constraint!