Iterated Dominance, Rationalizability, Correlated Equilibrium

CPSC 532A Lecture 7

October 3, 2006

Lecture Overview

Max-Min Strategies

- Player i's maxmin strategy is a strategy that maximizes i's worst-case payoff, in the situation where all the other players (whom we denote $-i$) happen to play the strategies which cause the greatest harm to i.
- The maxmin value (or safety level) of the game for player i is that minimum amount of payoff guaranteed by a maxmin strategy.
- Why would i want to play a maxmin strategy?

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- The maxmin value (or safety level) of the game for player i is that minimum amount of payoff guaranteed by a maxmin strategy.
- Why would i want to play a maxmin strategy?
  - a conservative agent maximizing worst-case payoff
  - a paranoid agent who believes everyone is out to get him

Definition

The maxmin strategy for player i is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_1, s_2)$, and the maxmin value for player i is $\max_{s_i} \min_{s_{-i}} u_i(s_1, s_2)$.

Minmax Theorem

Theorem (Minmax theorem (von Neumann, 1928))

In any finite, two-player, zero-sum game it is the case that:

1. *The maxmin value for one player is equal to the minmax value for the other player. By convention, the maxmin value for player 1 is called the value of the game.*
2. *For both players, the set of maxmin strategies coincides with the set of minmax strategies.*
3. *Any maxmin strategy profile (or, equivalently, minmax strategy profile) is a Nash equilibrium. Furthermore, these are all the Nash equilibria. Consequently, all Nash equilibria have the same payoff vector (namely, those in which player 1 gets the value of the game).*

Linear Programming

$$\begin{aligned} \text{maximize} \quad & \sum_i w_i x_i \\ \text{subject to} \quad & \sum_i w_i^c x_i \geq b^c && \forall c \in C \\ & x_i \geq 0 && \forall x_i \in X \end{aligned}$$

- These problems can be solved in polynomial time using interior point methods.
  - Interestingly, the (worst-case exponential) simplex method is often faster in practice.

Computing equilibria of zero-sum games

$$
\begin{aligned}
\text{minimize} \quad & U_1^* \\
\text{subject to} \quad & \sum_{a_2 \in A_2} u_1(a_1, a_2) \cdot s_2^{a_2} \le U_1^* && \forall a_1 \in A_1 \\
& \sum_{a_2 \in A_2} s_2^{a_2} = 1 \\
& s_2^{a_2} \ge 0 && \forall a_2 \in A_2
\end{aligned}
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- This program can also be used on a modified version of a general-sum two-player game to compute maxmin and minmax strategies.

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Domination

- Let s_i and s'_i be two strategies for player i, and let S_{-i} be is the set of all possible strategy profiles for the other players

Definition

s_i strictly dominates s'_i if $\forall s_{-i} \in S_{-i},\ u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$

Definition

s_i weakly dominates s'_i if $\forall s_{-i} \in S_{-i},\ u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ and $\exists s_{-i} \in S_{-i},\ u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$

Definition

s_i very weakly dominates s'_i if $\forall s_{-i} \in S_{-i},\ u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$

Lecture Overview

Recap

Iterated Removal

Computation

Rationalizability

Fun Game

Dominated strategies

- No equilibrium can involve a strictly dominated strategy
  - Thus we can remove it, and end up with a strategically equivalent game
  - This might allow us to remove another strategy that wasn't dominated before
  - Running this process to termination is called iterated removal of dominated strategies.

Iterated Removal of Dominated Strategies: Example

	L	C	R
U	$3, 1$	$0, 1$	$0, 0$
M	$1, 1$	$1, 1$	$5, 0$
D	$0, 1$	$4, 1$	$0, 0$

Iterated Removal of Dominated Strategies: Example

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- R is dominated by L.

Iterated Removal of Dominated Strategies: Example

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Iterated Removal of Dominated Strategies: Example

	L	C
U	$3, 1$	$0, 1$
M	$1, 1$	$1, 1$
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- M is dominated by the mixed strategy that selects U and D with equal probability.

Iterated Removal of Dominated Strategies: Example

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Iterated Removal of Dominated Strategies: Example

	L	C
U	$3, 1$	$0, 1$
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- No other strategies are dominated.

Iterated Removal of Dominated Strategies

- This process preserves Nash equilibria.
  - strict dominance: all equilibria preserved.
  - weak or very weak dominance: at least one equilibrium preserved.
- Thus, it can be used as a preprocessing step before computing an equilibrium
  - Some games are solvable using this technique.
  - Example: Traveler's Dilemma!
- What about the order of removal when there are multiple dominated strategies?
  - strict dominance: doesn't matter.
  - weak or very weak dominance: can affect which equilibria are preserved.

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Computational Problems in Domination

- Identifying strategies dominated by a pure strategy
- Identifying strategies dominated by a mixed strategy
- Identifying strategies that survive iterated elimination
- Asking whether a strategy survives iterated elimination under all elimination orderings
- We'll assume that i's utility function is strictly positive everywhere (why is this OK?)

Is s_i strictly dominated by any pure strategy?

for all pure strategies $a_i \in A_i$ for player i where $a_i \neq s_i$ **do**
  $dom \leftarrow true$
  for all pure strategy profiles $a_{-i} \in A_{-i}$ for the players other than i **do**
    if $u_i(s_i, a_{-i}) \geq u_i(a_i, a_{-i})$ **then**
      $dom \leftarrow false$
      break
    end if
  end for
  if $dom = true$ **then return** $true$
end for
return $false$

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end for
return $false$

- What is the complexity of this procedure?
- Why don't we have to check mixed strategies of the other players?
- What would we have to change to test for weak or very weak dominance?

Constraints for determining whether s_i is strictly dominated by any mixed strategy

$$\sum_{j \in A_i} p_j u_i(a_j, a_{-i}) > u_i(s_i, a_{-i}) \qquad \forall a_{-i} \in A_{-i}$$

$$p_j \geq 0 \qquad \forall j \in A_i$$

$$\sum_{j \in A_i} p_j = 1$$

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- What's wrong with this program?

Constraints for determining whether s_i is strictly dominated by any mixed strategy

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$$p_j \geq 0 \qquad \forall j \in A_i$$

$$\sum_{j \in A_i} p_j = 1$$

- What's wrong with this program?
  - strict inequality in the first constraint means we don't have an LP

LP for determining whether s_i is strictly dominated by any mixed strategy

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\text{minimize} \quad & \sum_{j \in A_i} p_j \\
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- This is clearly an LP. Why is it a solution to our problem?

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- This is clearly an LP. Why is it a solution to our problem?
  - if a solution exists with $\sum_j p_j < 1$ then we can add $1 - \sum_j p_j$ to some p_k and we'll have a dominating mixed strategy (since utility was assumed to be positive everywhere)
- Our original program (weak inequality) works for very weak domination
- For weak domination we can use that program with a different objective function trick.

Identifying strategies that survive iterated elimination

- This can be done by repeatedly solving our LPs: solving a polynomial number of LPs is still in $\mathcal{P}$.
  - Checking whether every pure strategy of every player is dominated by any other mixed strategy requires us to solve at worst $\sum_{i \in N} |A_i|$ linear programs.
  - Each step removes one pure strategy for one player, so there can be at most $\sum_{i \in N}(|A_i| - 1)$ steps.
  - Thus we need to solve $O((n \cdot a^*)^2)$ linear programs, where $a^* = \max_i |A_i|$.

Further questions about iterated elimination

1. **(Strategy Elimination)** Does there exist some elimination path under which the strategy s_i is eliminated?
2. **(Reduction Identity)** Given action subsets $A'_i \subseteq A_i$ for each player i, does there exist a maximally reduced game where each player i has the actions A'_i?
3. **(Uniqueness)** Does every elimination path lead to the same reduced game?
4. **(Reduction Size)** Given constants k_i for each player i, does there exist a maximally reduced game where each player i has exactly k_i actions?

Further questions about iterated elimination

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- For iterated strict dominance these problems are all in $\mathcal{P}$.
- For iterated weak or very weak dominance these problems are all $\mathcal{NP}$-complete.

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Rationalizability

Fun Game

Rationalizability

- Rather than ask what is irrational, ask what is a best response to some beliefs about the opponent
  - assumes opponent is rational
  - assumes opponent knows that you and the others are rational
  - ...
- Examples
  - is $heads$ rational in matching pennies?

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  - Yes, equilibrium strategies are always rationalizable.

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- Will there always exist a rationalizable strategy?
  - Yes, equilibrium strategies are always rationalizable.
- Furthermore, in two-player games, rationalizable $\Leftrightarrow$ survives iterated removal of strictly dominated strategies.

Lecture Overview

Fun game

	L	H	S
L	$90, 90$	$0, 0$	$0, 40$
B	$0, 0$	$180, 180$	$0, 40$

Fun game

	L	H	S
L	$90, 90$	$0, 0$	$400, 40$
B	$0, 0$	$180, 180$	$0, 40$

Fun game

	L	H	S
L	$90, 90$	$0, 0$	$0, 40;\ 400, 40$
B	$0, 0$	$180, 180$	$0, 40$

- What's the equilibrium?

Fun game

	L	H	S
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- What's the equilibrium?
  - 50-50 L-H dominates S for column, so we have a standard coordination game.

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- What's the equilibrium?
  - 50-50 L-H dominates S for column, so we have a standard coordination game.
- What happens when people play?
  - with $0, 40$, 96% row and 84% column choose the high payoff H, coordination occurs 80% of the time.
  - with $400, 40$, 64% row and 76% column chose H; coordination on H,H 32% of the time, coordination on L,L 16% of the time, uncoordinated over half the time