Multi-Good Auctions

CPSC 532A Lecture 23

November 30, 2006

Lecture Overview

1. Recap
2. Beyond IPV
3. Multiunit auctions
4. Combinatorial Auctions
5. Bidding Languages

Revenue Equivalence

- Which auction should an auctioneer choose? To some extent, it doesn't matter...

Theorem (Revenue Equivalence Theorem)

Assume that each of n risk-neutral agents has an independent private valuation for a single good at auction, drawn from a common cumulative distribution $F(v)$ that is strictly increasing and atomless on $[\underline{v}, \bar{v}]$. Then any auction mechanism in which

- *the good will be allocated to the agent with the highest valuation; and*
- *any agent with valuation $\underline{v}$ has an expected utility of zero;*

yields the same expected revenue, and hence results in any bidder with valuation v making the same expected payment.

Designing optimal auctions

Definition

Bidder i's virtual valuation is

$$\psi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

Definition

Bidder i's bidder-specific reserve price r_i^* is the value for which $\psi_i(r_i^*) = 0$.

Theorem

The optimal (single-good) auction is a sealed-bid auction in which every agent is asked to declare his valuation. The good is sold to the agent $i = \arg\max_i \psi_i(\hat{v}_i)$, as long as $v_i > r_i^*$. If the good is sold, the winning agent i is charged $\max(r_i^*, max_{j \neq i} \psi_j(v_j))$.

Going beyond IPV

- common value model
  - motivation: oil well
  - winner's curse
  - things can be improved by revealing more information
- general model
  - IPV + common value
  - example motivation: private value plus resale

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Affiliated Values

- Definition: a high value of one bidder's signal makes high values of other bidders' signals more likely
  - common value model is a special case
- generally, ascending auctions lead to higher expected prices than 2nd-price, which in turn leads to higher expected prices than 1st price
  - intuition: winner's gain depends on the privacy of his information.
  - The more the price paid depends on others' information (rather than expectations of others' information), the more closely this price is related to the winner's information, since valuations are affiliated
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- Linkage principle: if the seller has access to any private source of information which will be affiliated with the bidders' valuations, she should precommit to reveal it honestly.

Risk Attitudes

- Buyer:
  - no change under various risk attitudes for second price
  - in first-price, increasing bid amount increases probability of winning, decreases profit. This is good for risk-averse bidder, bad for risk-seeking bidder.
  - Risk averse, IPV: [First Price] $\succ$ [Japanese $=$ English $=$ Second]
  - Risk seeking, IPV: Second $\succ$ First
- Auctioneer:
  - revenue is fixed in first price auction (the expected amount of the 2nd-highest bid)
  - revenue varies in second price auction, with the same expected value
  - thus, a risk-averse seller prefers first-price to second-price.

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  - everyone should bid honestly in the final auction
  - bidder has an expected utility (conditioned on type) in that auction
  - in the second-last auction, bid the difference between valuation and the expected utility for losing (i.e., the expected utility for playing the second auction)
  - combining these last two auctions together, there's some expected utility to playing both of them
  - now this is the "expected utility of losing"
  - apply backward induction

Multiunit Auctions

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- how else can we sell the goods?
  - pay-your-bid: "discriminatory" pricing, because bidders will pay different amounts for the same thing
  - lowest winning bid: very similar to VCG, but ensures that bidders don't pay zero if there are fewer bids than units for sale
- in fact, the revenue equivalence theorem holds in this setting, so all these schemes must lead to the same expected payment.

Multiunit Valuations

How can bidders express their valuations in a multiunit auction?

- m homogeneous goods, let S denote some set
- general: let $p_1, \ldots, p_m$ be arbitrary, non-negative real numbers. Then $v(S) = \sum_{j=1}^{|S|} p_j$.
- downward sloping: general, but $p_1 \geq p_2 \geq \ldots \geq p_m$
- additive: $v(S) = c|S|$
- single-item: $v(S) = c$ if $s \neq \emptyset$; 0 otherwise
- fixed-budget: $v(S) = \min(c|S|, b)$
- majority: $v(S) = c$ if $|S| \geq m/2$, 0 otherwise

Lecture Overview

Valuations for heterogeneous goods

- now consider a case where multiple, heterogeneous goods are being sold.
- consider the sorts of valuations that agents could have in this case:
  - complementarity: for sets S and T, $v(S \cup T) > v(S) + v(T)$
    - e.g., a left shoe and a right shoe
  - substitutability: $v(S \cup T) < v(S) + v(T)$
    - e.g., two tickets to different movies playing at the same time
- substitutability is relatively easy to deal with
  - e.g., just sell the goods sequentially, or allow bid withdrawal
- complementarity is trickier...

Fun Game

1	2	3
4	5	6
7	8	9

- 9 plots of land for sale, geographically related as shown
- IPV, normally distributed with mean 50, stdev 5
- payoff:
  - if you get one good other than #5: v_i
  - any two goods: $3v_i$
  - any three (or more) goods: $5v_i$
- Rules:
  - auctioneer moves from one good to the next sequentially, holding an English auction for each good.
  - bidding stops on a good: move on to the next good
  - no bids for any of the 9 goods: end the auction

Combinatorial auctions

- running a simultaneous ascending auction is inefficient
  - exposure problem
  - inefficiency due to fear of exposure
- if we want an efficient outcome, why not just run VCG?
  - unfortunately, it requires solving an NP-hard problem
  - let there be n goods, m bids, sets C_j of XOR bids
  - weighted set packing problem:

$$\max \sum_{i=1}^{m} x_i p_i$$

$$\text{subject to} \sum_{i|g \in S_i} x_i \le 1 \qquad \forall g$$

$$x_i \in \{0,1\} \qquad \forall i$$

$$\sum_{k \in C_j} x_k \le 1 \qquad \forall j$$

Combinatorial auctions

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- we don't need the XOR constraints
  - instead, we can introduce "dummy goods" that don't correspond to goods in the auction, but that enforce XOR constraints.
  - amounts to exactly the same thing: the first constraint has the same form as the third

Lecture Overview

Expressing a bid in combinatorial auctions: OR bidding

- Atomic bid: (S, p) means $v(S) = p$
  - implicitly, an "AND" of the singletons in S
- OR bid: combine atomic bids
- let v_1, v_2 be arbitrary valuations

$$(v_1 \vee v_2)(S) = \max_{\substack{R,T \subseteq S \\ R \cup T = \emptyset}} [v_1(R) + v_2(S)]$$

Theorem

OR bids can express all valuations that do not have any substitutability, and only these valuations.

XOR Bids

- XOR bidding: allow substitutabilities
  - $(v_1 XOR v_2)(S) = \max(v_1(S), v_2(S))$

Theorem

XOR bids can represent any valuation

- this isn't really surprising, since we can enumerate valuations
- however, this implies that they don't represent everything efficiently

Theorem

Additive valuations require linear space with OR, exponential space with XOR

- likewise with many other valuations: any in which the price is different for every bundle

Composite Bidding Languages

- OR-of-XOR
- sets of XOR bids, where the bidder is willing to get either one or zero from each set
  - $(\ldots XOR \ldots XOR \ldots) OR (\ldots) OR (\ldots)$

Theorem

Any downward sloping valuation can be represented using the OR-of-XOR language using at most m^2 atomic bids.

- XOR-of-OR
  - a set of OR atomic bids, where the bidder is willing to select from only one of these sets
- generalized OR/XOR
  - arbitrary nesting of OR and XOR

The OR* Language

- OR*
  - OR, but uses dummy goods to simulate XOR constraints

Theorem

OR-of-XOR size $k \Rightarrow$ *OR* size* k, $\leq k$ *dummy goods*

Theorem

Generalized OR/XOR size $k \Rightarrow$ *OR* size* k, $\leq k^2$ *dummy goods*

Corollary

XOR-of-OR size $k \Rightarrow$ *OR* size* k, $\leq k^2$ *dummy goods*