Auction Theory I

CPSC 532A Lecture 20

November 23, 2006

Lecture Overview

VCG Caveats

- Frugality: VCG can undercharge agents arbitrarily
- Privacy: agents must declare all their private information
- Collusion: agents can gain
- Returning profits: very tricky

Some popular auctions

- English
- Dutch
- First-Price
- Second-Price

Some more exotic auction types

- Japanese auction
- All-pay auction
- Continuous double auction
- Call market ("periodic clear")

Continuous Double Auction

- every new order is matched as soon as it comes in, if possible
- otherwise, it goes on the order book
- this is how NASDAQ works

Call Market ("periodic clear")

- orders are matched periodically
- makes sense for settings where there is less liquidity
- this is used in e.g., the Arizona Stock Exchange

Lecture Overview

Intuitive comparison of 5 auctions

	English	Dutch	Japanese	1st-Price	2nd-Price
Duration	#bidders, increment	starting price, clock speed	#bidders, increment	fixed	fixed
Info Revealed	2nd-highest val; bounds on others	winner's bid	all val's but winner's	none	none
Jump bids	yes	n/a	no	n/a	n/a
Price Discovery	yes	no	yes	no	no

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- How should agents bid in these auctions?

Fun Game

- Valuation models:
  - the most important one: IPV
  - valuations are iid draws from some commonly-known distribution
    - do you see how we can write this as a Bayesian game?

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  - Dutch

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Price Discovery	yes	no	yes	no	no
Regret	no	yes	no	yes	no

Lecture Overview

Second-Price

Theorem

Truth-telling is a dominant strategy in a second-price auction.

- In fact, we know this already (do you see why?)
- However, we'll look at a simpler, direct proof.

Second-Price proof

Theorem

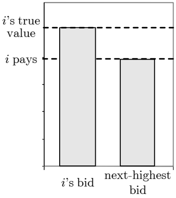
Truth-telling is a dominant strategy in a second-price auction.

Proof.

Assume that the other bidders bid in some arbitrary way. We must show that i's best response is always to bid truthfully. We'll break the proof into two cases:

1. Bidding honestly, i would win the auction
2. Bidding honestly, i would lose the auction

Second-Price proof (2)



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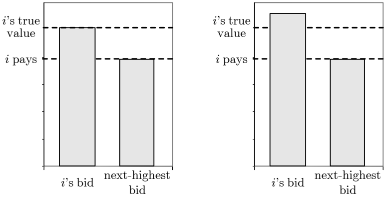

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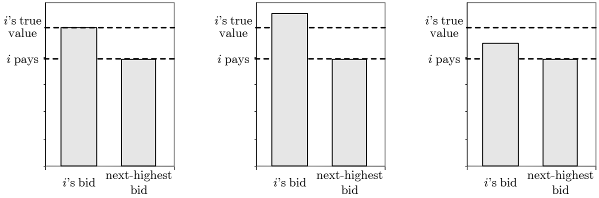

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- If i bids lower, he will either still win and still pay the same amount. . .

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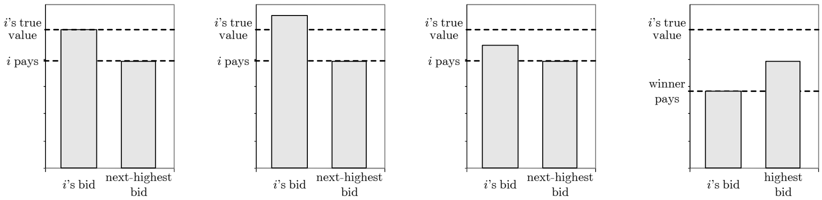
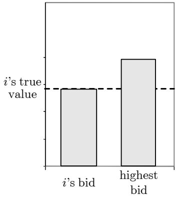


- Bidding honestly, i is the winner
- If i bids higher, he will still win and still pay the same amount
- If i bids lower, he will either still win and still pay the same amount. . . or lose and get utility of zero.

Second-Price proof (3)



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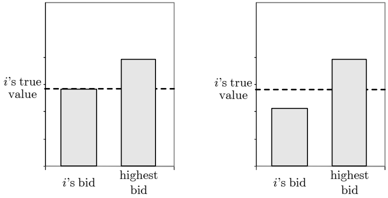

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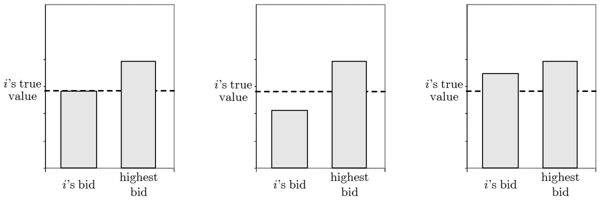

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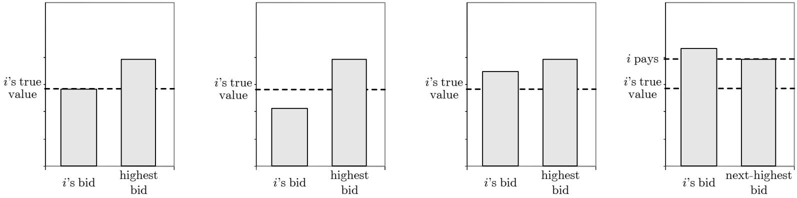

- Bidding honestly, i is not the winner
- If i bids lower, he will still lose and still pay nothing
- If i bids higher, he will either still lose and still pay nothing. . . or win and pay more than his valuation.

English and Japanese auctions

- A much more complicated strategy space
  - extensive form game
  - bidders are able to condition their bids on information revealed by others
  - in the case of English auctions, the ability to place jump bids
- intuitively, though, the revealed information doesn't make any difference in the IPV setting.

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Theorem

Under the independent private values model (IPV), it is a dominant strategy for bidders to bid up to (and not beyond) their valuations in both Japanese and English auctions.

Lecture Overview

First-Price and Dutch

Theorem

First-Price and Dutch auctions are strategically equivalent.

- In both first-price and Dutch, a bidder must decide on the amount he's willing to pay, conditional on having placed the highest bid.
  - despite the fact that Dutch auctions are extensive-form games, the only thing a winning bidder knows about the others is that all of them have decided on lower bids
    - e.g., he does not know *what* these bids are
    - this is exactly the thing that a bidder in a first-price auction assumes when placing his bid anyway.
- Note that this is a stronger result than the connection between second-price and English.

Discussion

- So, why are both auction types held in practice?
  - First-price auctions can be held asynchronously
  - Dutch auctions are fast, and require minimal communication: only one bit needs to be transmitted from the bidders to the auctioneer.
- How should bidders bid in these auctions?

Discussion

- So, why are both auction types held in practice?
  - First-price auctions can be held asynchronously
  - Dutch auctions are fast, and require minimal communication: only one bit needs to be transmitted from the bidders to the auctioneer.
- How should bidders bid in these auctions?
  - They should clearly bid less than their valuations.
  - There's a tradeoff between:
    - probability of winning
    - amount paid upon winning
  - Bidders don't have a dominant strategy anymore.

Analysis

Theorem

In a first-price auction with two risk-neutral bidders whose valuations are drawn independently and uniformly at random from $[0, 1]$, $(\frac{1}{2}v_1, \frac{1}{2}v_2)$ *is a Bayes-Nash equilibrium strategy profile.*

Proof.

Assume that bidder 2 bids $\frac{1}{2}v_2$, and bidder 1 bids s_1. From the fact that v_2 was drawn from a uniform distribution, all values of v_2 between 0 and 1 are equally likely. Bidder 1's expected utility is

$$E[u_1] = \int_0^1 u_1 dv_2. \tag{1}$$

Note that the integral in Equation (1) can be broken up into two smaller integrals that differ on whether or not player 1 wins the auction.

$$E[u_1] = \int_0^{2s_1} u_1 dv_2 + \int_{2s_1}^1 u_1 dv_2$$

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Proof.

We can now substitute in values for u_1. In the first case, because 2 bids $\frac{1}{2}v_2$, 1 wins when $v_2 < 2s_1$, and gains utility $v_1 - s_1$. In the second case 1 loses and gains utility 0. Observe that we can ignore the case where the agents have the same valuation, because this occurs with probability zero.

$$E[u_1] = \int_0^{2s_1} (v_1 - s_1)dv_2 + \int_{2s_1}^{1} (0)dv_2$$
$$= (v_1 - s_1)v_2\Big|_0^{2s_1}$$
$$= 2v_1s_1 - 2s_1^2 \qquad (2)$$

Analysis

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In a first-price auction with two risk-neutral bidders whose valuations are drawn independently and uniformly at random from $[0, 1]$*,* $(\frac{1}{2}v_1, \frac{1}{2}v_2)$ *is a Bayes-Nash equilibrium strategy profile.*

Proof.

We can find bidder 1's best response to bidder 2's strategy by taking the derivative of Equation (2) and setting it equal to zero:

$$\frac{\partial}{\partial s_1}(2v_1 s_1 - 2s_1^2) = 0$$

$$2v_1 - 4s_1 = 0$$

$$s_1 = \frac{1}{2}v_1$$

Thus when player 2 is bidding half her valuation, player 1's best strategy is to bid half his valuation. The calculation of the optimal bid for player 2 is analogous, given the symmetry of the game and the equilibrium.

More than two bidders

- Very narrow result: two bidders, uniform valuations.
- Still, first-price auctions are not incentive compatible
  - hence, unsurprisingly, not equivalent to second-price auctions

Theorem

In a first-price sealed bid auction with n risk-neutral agents whose valuations are independently drawn from a uniform distribution on the same bounded interval of the real numbers, the unique symmetric equilibrium is given by the strategy profile $\left(\frac{n-1}{n}v_1, \ldots, \frac{n-1}{n}v_n\right)$.

- proven using a similar argument, but more involved calculus
- a broader problem: that proof only showed how to *verify* an equilibrium strategy.
  - How do we identify one in the first place?

Lecture Overview

Revenue Equivalence

- Which auction should an auctioneer choose? To some extent, it doesn't matter...

Theorem (Revenue Equivalence Theorem)

Assume that each of n risk-neutral agents has an independent private valuation for a single good at auction, drawn from a common cumulative distribution $F(v)$ that is strictly increasing and atomless on $[\underline{v}, \bar{v}]$. Then any auction mechanism in which

- *the good will be allocated to the agent with the highest valuation; and*
- *any agent with valuation $\underline{v}$ has an expected utility of zero;*

yields the same expected revenue, and hence results in any bidder with valuation v making the same expected payment.