VCG

CPSC 532A Lecture 19

November 16, 2006
Lecture Overview

1. Recap
2. Groves Uniqueness
3. VCG
4. VCG example
5. Individual Rationality
6. Budget Balance
Risk Neutrality

(a) Risk neutrality

(b) Risk neutrality: fair lottery

Figure 8.3
Risk attitudes: Risk aversion, risk neutrality, risk seeking, and in each case, utility for the outcomes of a fair lottery.

$c^\text{c⃝} \text{Shoham and Leyton-Brown, 2006}$
Risk Aversion

(c) Risk aversion

(d) Risk aversion: fair lottery
Quasilinear Mechanisms

Definition (Direct quasilinear mechanism)

A *direct quasilinear mechanism* (over a set of agents $N$ and a set of outcomes $O = X \times \mathbb{R}^n$) is a pair $(x, p)$. It defines a standard mechanism in the quasilinear setting, where for each $i$, $A_i = \Theta_i$.

- An agent's valuation for choice $x \in X$: $v_i(x) = u_i(x, \theta)$
  - the maximum amount $i$ would be willing to pay to get $x$
  - in fact, $i$ would be indifferent between keeping the money and getting $x$
- Equivalent definition: mechanisms that ask agents $i$ to declare $v_i(x)$ for each $x \in X$
- Define $\hat{v}_i$ as the valuation that agent $i$ declares to such a direct mechanism
  - may be different from his true valuation $v_i$
- Also define the tuples $\hat{v}, \hat{v}_{-i}$
Truthfulness

Definition (Truthfulness)
A mechanism is *truthful* if $\forall i \forall v_i$, agent $i$’s equilibrium strategy is to adopt the strategy $\hat{v}_i = v_i$.

- Our definition before, adapted for the quasilinear setting
Efficiency

Definition (Efficiency)

A mechanism is efficient if it selects a choice $x$ such that
\[
\forall i \forall v_i \forall x', \sum_i v_i(x) \geq \sum_i v_i(x').
\]

- An efficient mechanism selects the choice that maximizes the sum of agents’ utilities, disregarding monetary payments.
- Called economic efficiency to distinguish from other (e.g., computational) notions.
- Also called social-welfare maximization.
- Note: defined in terms of true (not declared) valuations, not declared valuations.
Definition (Budget balance)

A mechanism is \textbf{budget balanced} when \( \forall \hat{v}, \sum_i p_i(\hat{v}) = 0 \). 

- regardless of the agents' types, the mechanism collects and disburses the same amount of money from and to the agents
- relaxed version: \textbf{weak budget balance}: \( \forall \hat{v} \sum_i p_i(\hat{v}) \geq 0 \)
  - the mechanism never takes a loss, but it may make a profit
- \textbf{Budget balance} can be required to hold \textit{ex ante}:
  \[ \mathbb{E}_v \sum_i p_i(v) = 0 \]
  - the mechanism must break even or make a profit only on expectation
Individual-Rationality

**Definition (Ex-interim individual rationality)**

A mechanism is **ex-interim individual rational** when
\[
\forall i \forall v_i, \mathbb{E}_{v_{-i}|v_i} v_i(x(s_i(v_i), s_{-i}(v_{-i}))) - p_i(s_i(v_i), s_{-i}(v_{-i})) \geq 0,
\]
where \( s \) is the equilibrium strategy profile.

- no agent loses by participating in the mechanism.
- **ex-interim** because it holds for every possible valuation for agent \( i \), but averages over the possible valuations of the other agents.

**Definition (Ex-post individual rationality)**

A mechanism is **ex-post individual rational** when
\[
\forall i \forall v, v_i(x(s(v))) - p_i(s(v)) \geq 0,
\]
where \( s \) is the equilibrium strategy profile.
The Groves Mechanism

Definition (Groves mechanism)

The Groves mechanism is a direct quasilinear mechanism \((\mathbb{R}^X|^n, \chi, p)\), where

\[
\chi(\hat{v}) = \text{arg max}_x \sum_i \hat{v}_i(x)
\]

\[
p_i(\hat{v}) = h_i(\hat{v}_{-i}) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))
\]

Theorem

Truth telling is a dominant strategy under the Groves mechanism.
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Theorem

An efficient social choice function $C : \mathbb{R}^{X^n} \rightarrow X \times \mathbb{R}^n$ can be implemented in dominant strategies for agents with unrestricted quasilinear utilities only if $p_i(v) = h(v_{-i}) - \sum_{j \neq i} v_j(\mathcal{X}(v))$.

- it turns out that the same result also holds for the broader class of Bayes-Nash incentive-compatible efficient mechanisms.
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Clarke Tax

**Definition (Clarke tax)**

The Clarke tax sets the $h_i$ term in a Groves mechanism as

$$h_i(\hat{v}_-i) = \sum_{j \neq i} \hat{v}_j \left( \chi(\hat{v}_-i) \right),$$

where $\chi$ is the Groves mechanism allocation function.
Definition (Vickrey-Clarke-Groves (VCG) mechanism)

The *Vickrey-Clarke-Groves mechanism* is a direct quasilinear mechanism \((\mathbb{R}^{|X|^n}, \chi, p)\), where

\[
\chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x)
\]

\[
p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}-i)) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))
\]
VCG discussion

\[ \chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x) \]

\[ p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}_{-i})) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \]

- You get paid everyone’s utility under the allocation that is actually chosen
  - except your own, but you get that directly as utility
- Then you get charged everyone’s utility in the world where you don’t participate
- Thus you pay your social cost
VCG discussion

\[ \chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x) \]

\[ p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(x(\hat{v}_i)) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \]

Questions:
- who pays 0?
- agents who don't affect the outcome
- who pays more than 0? (pivotal) agents who make things worse for others by existing
- who gets paid? (pivotal) agents who make things better for others by existing
VCG discussion

\[ x(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x) \]

\[ p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(x(\hat{v}_i)) - \sum_{j \neq i} \hat{v}_j(x(\hat{v})) \]

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VCG properties

\[ \chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x) \]

\[ p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}_-i)) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \]

- Because only pivotal agents have to pay, VCG is also called the pivot mechanism.
- It’s dominant strategy truthful, because it’s a Groves mechanism.
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Selfish routing example

What outcome will be selected by $\chi$?
Selfish routing example

What outcome will be selected by $\chi$? path $ABEF$. 

First, note that because the Clarke tax does not depend on an agent $i$’s own declaration $v_i$, our previous arguments that Groves mechanisms are dominant strategy truthful and efficient transfer immediately to the VCG mechanism. Now, we’ll try to provide some intuition about the VCG payment rule. Assume that all agents follow their dominant strategies and declare their valuations truthfully. The second sum in the VCG payment rule pays each agent $i$ the sum of every other agent $j \neq i$’s utility for the mechanism’s choice. The first sum charges each agent $i$ the sum of every other agent’s utility for the choice that would have been made had $i$ not participated in the mechanism. Thus, each agent is made to pay his social cost—the aggregate impact that his participation has on other agents’ utilities.

What can we say about the amounts of different agents’ payments to the mechanism? If some agent $i$ does not change the mechanism’s choice by his participation—that is, if $x(v) = x(v - i)$—then the two sums in the VCG payment function will cancel out. The social cost of $i$’s participation is zero, and so he has to pay nothing. In order for an agent $i$ to be made to pay a nonzero amount, he must be pivotal in the sense that the mechanism’s choice $x(v)$ is different from its choice without $i$, $x(v - i)$. This is why VCG is sometimes called the pivot mechanism—only pivotal agents are made to pay. Of course, it’s possible that some agents will improve other agents’ utility by participating; such agents will be made to pay a negative amount, or in other words will be paid by the mechanism.

Let’s see an example of how the VCG mechanism works. Recall that Section 8.1.2 discussed the problem of selfish routing in a transportation network. We’ll now reconsider that example, and determine what route and what payments the VCG mechanism would select. For convenience, we reproduce Figure 8.1 as Figure 8.4, and label the nodes so that we have names to refer to the agents (the edges).
Selfish routing example

- What outcome will be selected by $\chi$? path $ABEF$.
- How much will $AC$ have to pay?
Selfish routing example

- What outcome will be selected by \( \chi \)? path \( ABEF \).
- How much will \( AC \) have to pay?
  - The shortest path taking his declaration into account has a length of 5, and imposes a cost of \(-5\) on agents other than him (because it does not involve him). Likewise, the shortest path without \( AC \)'s declaration also has a length of 5. Thus, his payment \( p_{AC} = (-5) - (-5) = 0 \).
  - This is what we expect, since \( AC \) is not pivotal.
  - Likewise, \( BD, CE, CF \) and \( DF \) will all pay zero.
Selfish routing example

How much will $AB$ pay?

The shortest path taking $AB$'s declaration into account has a length of 5, and imposes a cost of 2 on other agents. The shortest path without $AB$ is $ACEF$, which has a cost of 6. Thus $p_{AB} = (−6) − (−2) = −4$.
Selfish routing example

- How much will $AB$ pay?
  - The shortest path taking $AB$’s declaration into account has a length of 5, and imposes a cost of 2 on other agents.
  - The shortest path without $AB$ is $ACEF$, which has a cost of 6.
  - Thus $p_{AB} = (-6) - (-2) = -4$. 

![Transportation network with selfish agents.](image)
Recap

Groves Uniqueness

VCG

VCG example

Individual Rationality

Budget Balance

Selfish routing example

First, note that because the Clarke tax does not depend on an agent $i$'s own declaration $\hat{v}_i$, our previous arguments that Groves mechanisms are dominant strategy truthful and efficient transfer immediately to the VCG mechanism. Now, we'll try to provide some intuition about the VCG payment rule. Assume that all agents follow their dominant strategies and declare their valuations truthfully. The second sum in the VCG payment rule pays each agent $i$ the sum of every other agent $j \neq i$'s utility for the mechanism's choice. The first sum charges each agent $i$ the sum of every other agent's utility for the choice that would have been made had $i$ not participated in the mechanism. Thus, each agent is made to pay his social cost—the aggregate impact that his participation has on other agents' utilities.

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![Figure 8.4 Transportation network with selfish agents.](c⃝ Shoham and Leyton-Brown, 2006)

How much will $BE$ pay?

$$p_{BE} = (−6) − (−4) = −2.$$

How much will $EF$ pay?

$$p_{EF} = (−7) − (−4) = −3.$$

$EF$ and $BE$ have the same costs but are paid different amounts. Why?

$EF$ has more market power: for the other agents, the situation without $EF$ is worse than the situation without $BE$. 

\[ \text{Figure 8.4} \text{ Transportation network with selfish agents.} \]

\[ \text{© Shoham and Leyton-Brown, 2006} \]
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![Transportation network with selfish agents.](image)

- How much will $BE$ pay? $p_{BE} = (-6) - (-4) = -2$. 

...
Selfish routing example

- How much will $BE$ pay? $p_{BE} = (-6) - (-4) = -2$.
- How much will $EF$ pay?

![Transportation network with selfish agents.](image-url)

\[\begin{align*}
&\text{How much will } BE \text{ pay? } p_{BE} = (-6) - (-4) = -2. \\
&\text{How much will } EF \text{ pay?}
\end{align*}\]
Selfish routing example

How much will $BE$ pay? $p_{BE} = (-6) - (-4) = -2$.

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Selfish routing example

How much will $BE$ pay? $p_{BE} = (-6) - (-4) = -2$.

How much will $EF$ pay? $p_{EF} = (-7) - (-4) = -3$.

$EF$ and $BE$ have the same costs but are paid different amounts. Why?
Selfish routing example

- How much will $BE$ pay? $p_{BE} = (-6) - (-4) = -2$.
- How much will $EF$ pay? $p_{EF} = (-7) - (-4) = -3$.
  - $EF$ and $BE$ have the same costs but are paid different amounts. Why?
  - $EF$ has more market power: for the other agents, the situation without $EF$ is worse than the situation without $BE$.
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Two definitions

**Definition (Choice-set monotonicity)**

An environment exhibits **choice-set monotonicity** if
\[ \forall i, |X_{-i}| \leq |X| . \]

- removing any agent weakly decreases—that is, never increases—the mechanism’s set of possible choices \( X \)

**Definition (No negative externalities)**

An environment exhibits **no negative externalities** if
\[ \forall i \forall x \in X_{-i}, v_i(x) \geq 0. \]

- every agent has zero or positive utility for any choice that can be made without his participation
Example: road referendum

Consider the problem of holding a referendum to decide whether or not to build a road.

- The set of choices is independent of the number of agents, satisfying choice-set monotonicity.
- No agent negatively values the project, though some might value the situation in which the project is not undertaken more highly than the situation in which it is.
Example: simple exchange

Example

Consider a market setting consisting of agents interested in buying a single unit of a good such as a share of stock, and another set of agents interested in selling a single unit of this good. The choices in this environment are sets of buyer-seller pairings (prices are imposed through the payment function).

- If a new agent is introduced into the market, no previously-existing pairings become infeasible, but new ones become possible; thus choice-set monotonicity is satisfied.
- Because agents have zero utility both for choices that involve trades between other agents and no trades at all, there are no negative externalities.
Theorem

The VCG mechanism is ex-post individual rational when the choice set monotonicity and no negative externalities properties hold.

Proof.

All agents truthfully declare their valuations in equilibrium. Then

\[ u_i = v_i(\chi(v)) - \left( \sum_{j \neq i} v_j(\chi(v_i)) - \sum_{j \neq i} v_j(\chi(v)) \right) \]

\[ = \sum_i v_i(\chi(v)) - \sum_{j \neq i} v_j(\chi(v_i)) \]  

(1)

\( \chi(v) \) is the outcome that maximizes social welfare, and that this optimization could have picked \( \chi(v_i) \) instead (by choice set monotonicity). Thus,

\[ \sum_j v_j(\chi(v)) \geq \sum_j v_j(\chi(v_i)). \]
Theorem

The VCG mechanism is ex-post individual rational when the choice set monotonicity and no negative externalities properties hold.

Proof.

\[ \sum_j v_j(x(v)) \geq \sum_j v_j(x(v_{-i})). \]

Furthermore, from no negative externalities,

\[ v_i(x(v_{-i})) \geq 0. \]

Therefore,

\[ \sum_i v_i(x(v)) \geq \sum_{j \neq i} v_j(x(v_{-i})), \]

and thus Equation (1) is non-negative.
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Another property

**Definition (No single-agent effect)**

An environment exhibits **no single-agent effect** if \( \forall x, \forall i \) such that \( \exists v_{-i} \) where \( x \in \arg \max \sum_{j} v_{j}(x) \) there exists a choice \( x' \) that is feasible without \( i \) and that has \( \sum_{j \neq i} v_{j}(x') \geq \sum_{j \neq i} v_{j}(x) \).

**Example**

Consider a single-sided auction. Dropping an agent just reduces the amount of competition, making the others better off.
Good news

Theorem

The VCG mechanism is weakly budget-balanced when the no single-agent effect property holds.

Proof.

Assume truth-telling in equilibrium. We must show that the sum of transfers from agents to the center is greater than or equal to zero.

\[
\sum_i p_i(v) = \sum_i \left( \sum_{j \neq i} v_j(\chi(v_i)) - \sum_{j \neq i} v_j(\chi(v)) \right)
\]

From the no single-agent effect condition we have that

\[
\forall i \sum_{j \neq i} v_j(\chi(v_i)) \geq \sum_{j \neq i} v_j(\chi(v))
\]

Thus the result follows directly.
Bad news

Theorem

No dominant strategy incentive-compatible mechanism is always both efficient and weakly budget balanced, even if agents are restricted to the simple exchange setting.

Theorem

No Bayes-Nash incentive-compatible mechanism is always simultaneously efficient, weakly budget balanced and ex-interim individual rational, even if agents are restricted to quasilinear utility functions.