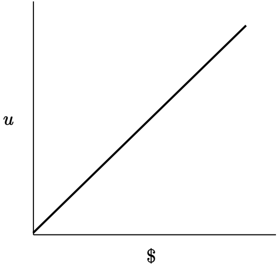
VCG

CPSC 532A Lecture 19

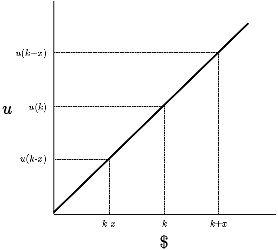
November 16, 2006

Lecture Overview

Risk Neutrality

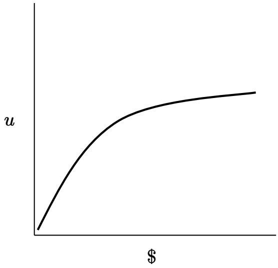


(a) Risk neutrality

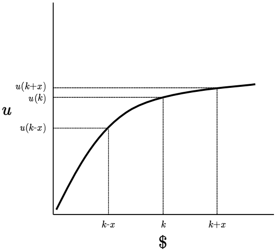


(b) Risk neutrality: fair lottery

Risk Aversion



(c) Risk aversion



(d) Risk aversion: fair lottery

Quasilinear Mechanisms

Definition (Direct quasilinear mechanism)

A *direct quasilinear mechanism* (over a set of agents N and a set of outcomes $O = X \times \mathbb{R}^n$) is a pair (χ, p). It defines a standard mechanism in the quasilinear setting, where for each i, $A_i = \Theta_i$.

- An agent's valuation for choice $x \in X$: $v_i(x) = u_i(x, \theta)$
  - the maximum amount i would be willing to pay to get x
  - in fact, i would be indifferent between keeping the money and getting x
- Equivalent definition: mechanisms that ask agents i to declare $v_i(x)$ for each $x \in X$
- Define $\hat{v}_i$ as the valuation that agent i declares to such a direct mechanism
  - may be different from his true valuation v_i
- Also define the tuples $\hat{v}$, $\hat{v}_{-i}$

Truthfulness

Definition (Truthfulness)

A mechanism is *truthful* if $\forall i \forall v_i$, agent i's equilibrium strategy is to adopt the strategy $\hat{v}_i = v_i$.

- Our definition before, adapted for the quasilinear setting

Efficiency

Definition (Efficiency)

A mechanism is efficient if it selects a choice x such that $\forall i \forall v_i \forall x', \sum_i v_i(x) \geq \sum_i v_i(x')$.

- An efficient mechanism selects the choice that maximizes the sum of agents' utilities, disregarding monetary payments.
- Called economic efficiency to distinguish from other (e.g., computational) notions
- Also called social-welfare maximization
- Note: defined in terms of true (not declared) valuations, not declared valuations.

Budget Balance

Definition (Budget balance)

A mechanism is budget balanced when $\forall \hat{v}, \ \sum_i p_i(\hat{v}) = 0$.

- regardless of the agents' types, the mechanism collects and disburses the same amount of money from and to the agents
- relaxed version: weak budget balance: $\forall \hat{v} \ \sum_i p_i(\hat{v}) \geq 0$
  - the mechanism never takes a loss, but it may make a profit
- Budget balance can be required to hold *ex ante*: $\mathbb{E}_v \sum_i p_i(v) = 0$
  - the mechanism must break even or make a profit only on expectation

Individual-Rationality

Definition (*Ex-interim* individual rationality)

A mechanism is ex-interim individual rational when $\forall i \forall v_i,\ \mathbb{E}_{v_{-i}|v_i} v_i(\chi(s_i(v_i), s_{-i}(v_{-i}))) - p_i(s_i(v_i), s_{-i}(v_{-i})) \geq 0$, where s is the equilibrium strategy profile.

- no agent loses by participating in the mechanism.
- *ex-interim* because it holds for *every* possible valuation for agent i, but averages over the possible valuations of the other agents.

Definition (*Ex-post* individual rationality)

A mechanism is ex-post individual rational when $\forall i \forall v,\ v_i(\chi(s(v))) - p_i(s(v)) \geq 0$, where s is the equilibrium strategy profile.

The Groves Mechanism

Definition (Groves mechanism)

The Groves mechanism is a direct quasilinear mechanism $(\mathbb{R}^{|X|n}, \chi, p)$, where

$$\chi(\hat{v}) = \arg\max_x \sum_i \hat{v}_i(x)$$

$$p_i(\hat{v}) = h_i(\hat{v}_{-i}) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

Theorem

Truth telling is a dominant strategy under the Groves mechanism.

Lecture Overview

Groves Uniqueness

Theorem

An efficient social choice function $C : \mathbb{R}^{Xn} \to X \times \mathbb{R}^n$ *can be implemented in dominant strategies for agents with unrestricted quasilinear utilities only if* $p_i(v) = h(v_{-i}) - \sum_{j \neq i} v_j(\chi(v))$.

- it turns out that the same result also holds for the broader class of Bayes-Nash incentive-compatible efficient mechanisms.

Lecture Overview

Clarke Tax

Definition (Clarke tax)

The Clarke tax sets the h_i term in a Groves mechanism as

$$h_i(\hat{v}_{-i}) = \sum_{j \neq i} \hat{v}_j \left(\chi(\hat{v}_{-i})\right),$$

where χ is the Groves mechanism allocation function.

VCG

Definition (Vickrey-Clarke-Groves (VCG) mechanism)

The *Vickrey-Clarke-Groves mechanism* is a direct quasilinear mechanism $(\mathbb{R}^{|X|n}, \chi, p)$, where

$$\chi(\hat{v}) = \arg\max_x \sum_i \hat{v}_i(x)$$

$$p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j\left(\chi(\hat{v}_{-i})\right) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

VCG discussion

$$\chi(\hat{v}) = \arg\max_x \sum_i \hat{v}_i(x)$$

$$p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j\left(\chi(\hat{v}_{-i})\right) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

- You get paid everyone's utility under the allocation that is actually chosen
  - except your own, but you get that directly as utility
- Then you get charged everyone's utility in the world where you don't participate
- Thus you pay your social cost

VCG discussion

$$\chi(\hat{v}) = \arg\max_x \sum_i \hat{v}_i(x)$$

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Questions:

- who pays 0?

VCG discussion

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Questions:

- who pays 0?
  - agents who don't affect the outcome

VCG discussion

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Questions:

- who pays 0?
    - agents who don't affect the outcome
- who pays more than 0?

VCG discussion

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- who pays more than 0?
  - (pivotal) agents who make things worse for others by existing

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- who gets paid?

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- who gets paid?
  - (pivotal) agents who make things better for others by existing

VCG properties

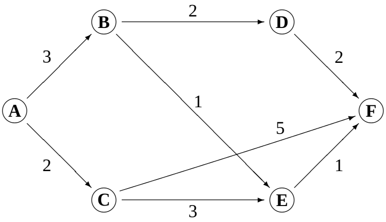
$$\chi(\hat{v}) = \arg\max_x \sum_i \hat{v}_i(x)$$

$$p_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j\left(\chi(\hat{v}_{-i})\right) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

- Because only pivotal agents have to pay, VCG is also called the pivot mechanism
- It's dominant strategy truthful, because it's a Groves mechanism

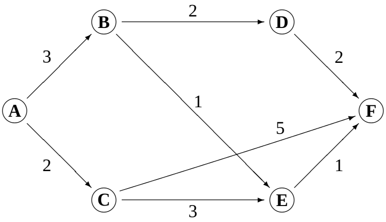
Lecture Overview

Selfish routing example



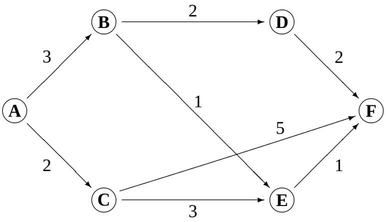
- What outcome will be selected by χ?

Selfish routing example



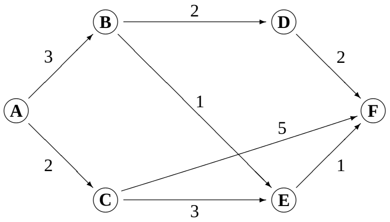
- What outcome will be selected by χ? path $ABEF$.

Selfish routing example



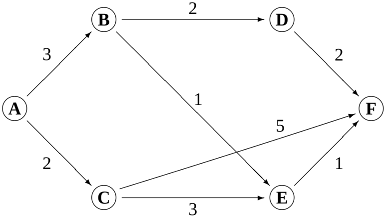
- What outcome will be selected by χ? path $ABEF$.
- How much will AC have to pay?

Selfish routing example



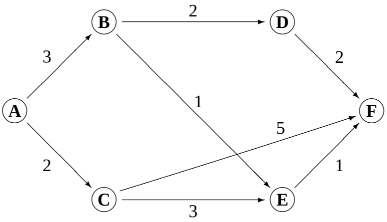
- What outcome will be selected by χ? path $ABEF$.
- How much will AC have to pay?
  - The shortest path taking his declaration into account has a length of 5, and imposes a cost of -5 on agents other than him (because it does not involve him). Likewise, the shortest path without AC's declaration also has a length of 5. Thus, his payment $p_{AC} = (-5) - (-5) = 0$.
  - This is what we expect, since AC is not pivotal.
  - Likewise, BD, CE, CF and DF will all pay zero.

Selfish routing example



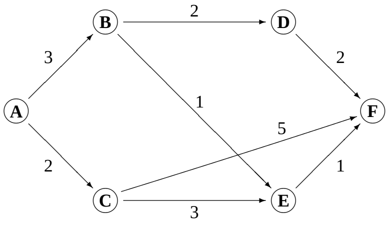
- How much will AB pay?

Selfish routing example



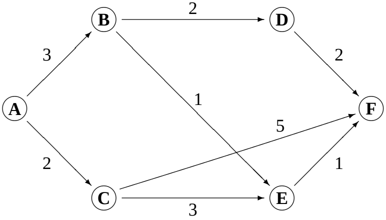
- How much will AB pay?
  - The shortest path taking AB's declaration into account has a length of 5, and imposes a cost of 2 on other agents.
  - The shortest path without AB is $ACEF$, which has a cost of 6.
  - Thus $p_{AB} = (-6) - (-2) = -4$.

Selfish routing example



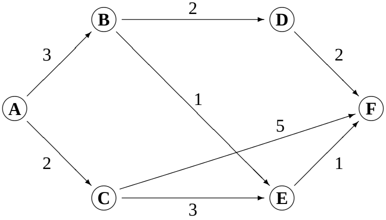
- How much will BE pay?

Selfish routing example



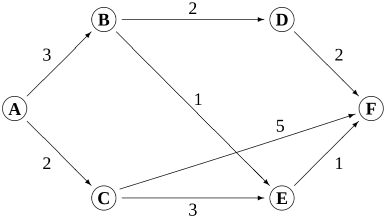
- How much will BE pay? $p_{BE} = (-6) - (-4) = -2$.

Selfish routing example



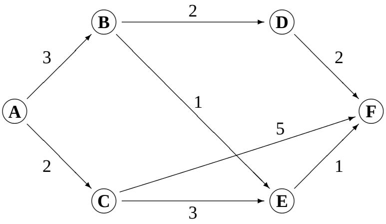
- How much will BE pay? $p_{BE} = (-6) - (-4) = -2$.
- How much will EF pay?

Selfish routing example



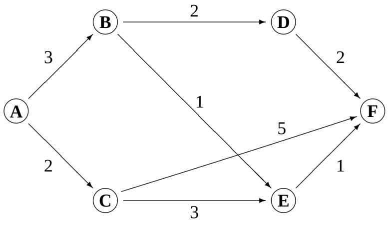
- How much will BE pay? $p_{BE} = (-6) - (-4) = -2.$
- How much will EF pay? $p_{EF} = (-7) - (-4) = -3.$

Selfish routing example



- How much will BE pay? $p_{BE} = (-6) - (-4) = -2$.
- How much will EF pay? $p_{EF} = (-7) - (-4) = -3$.
    - EF and BE have the same costs but are paid different amounts. Why?

Selfish routing example



- How much will BE pay? $p_{BE} = (-6) - (-4) = -2$.
- How much will EF pay? $p_{EF} = (-7) - (-4) = -3$.
  - EF and BE have the same costs but are paid different amounts. Why?
  - EF has more *market power*: for the other agents, the situation without EF is worse than the situation without BE.

Lecture Overview

Two definitions

Definition (Choice-set monotonicity)

An environment exhibits choice-set monotonicity if $\forall i, |X_{-i}| \leq |X|$.

- removing any agent weakly decreases—that is, never increases—the mechanism's set of possible choices X

Definition (No negative externalities)

An environment exhibits no negative externalities if $\forall i \forall x \in X_{-i}, v_i(x) \geq 0$.

- every agent has zero or positive utility for any choice that can be made without his participation

Example: road referendum

Example

Consider the problem of holding a referendum to decide whether or not to build a road.

- The set of choices is independent of the number of agents, satisfying choice-set monotonicity.
- No agent negatively values the project, though some might value the situation in which the project is not undertaken more highly than the situation in which it is.

Example: simple exchange

Example

Consider a market setting consisting of agents interested in buying a single unit of a good such as a share of stock, and another set of agents interested in selling a single unit of this good. The choices in this environment are sets of buyer-seller pairings (prices are imposed through the payment function).

- If a new agent is introduced into the market, no previously-existing pairings become infeasible, but new ones become possible; thus choice-set monotonicity is satisfied.
- Because agents have zero utility both for choices that involve trades between other agents and no trades at all, there are no negative externalities.

VCG Individual Rationality

Theorem

The VCG mechanism is ex-post individual rational when the choice set monotonicity and no negative externalities properties hold.

Proof.

All agents truthfully declare their valuations in equilibrium. Then

$$u_i = v_i(\chi(v)) - \left(\sum_{j \neq i} v_j(\chi(v_{-i})) - \sum_{j \neq i} v_j(\chi(v)) \right)$$

$$= \sum_i v_i(\chi(v)) - \sum_{j \neq i} v_j(\chi(v_{-i})) \quad (1)$$

$\chi(v)$ is the outcome that maximizes social welfare, and that this optimization could have picked $\chi(v_{-i})$ instead (by choice set monotonicity). Thus,

$$\sum_j v_j(\chi(v)) \geq \sum_j v_j(\chi(v_{-i})).$$

VCG Individual Rationality

Theorem

The VCG mechanism is ex-post individual rational when the choice set monotonicity and no negative externalities properties hold.

Proof.

$$\sum_j v_j(\chi(v)) \geq \sum_j v_j(\chi(v_{-i})).$$

Furthermore, from no negative externalities,

$$v_i(\chi(v_{-i})) \geq 0.$$

Therefore,

$$\sum_i v_i(\chi(v)) \geq \sum_{j \neq i} v_j(\chi(v_{-i})),$$

and thus Equation (1) is non-negative. □

Lecture Overview

Another property

Definition (No single-agent effect)

An environment exhibits no single-agent effect if $\forall x, \forall i$ such that $\exists v_{-i}$ where $x \in \arg\max \sum_j v_j(x)$ there exists a choice x' that is feasible without i and that has $\sum_{j \neq i} v_j(x') \geq \sum_{j \neq i} v_j(x)$.

Example

Consider a single-sided auction. Dropping an agent just reduces the amount of competition, making the others better off.

Good news

Theorem

The VCG mechanism is weakly budget-balanced when the no single-agent effect property holds.

Proof.

Assume truth-telling in equilibrium. We must show that the sum of transfers from agents to the center is greater than or equal to zero.

$$\sum_i p_i(v) = \sum_i \left(\sum_{j \neq i} v_j(\chi(v_{-i})) - \sum_{j \neq i} v_j(\chi(v)) \right)$$

From the no single-agent effect condition we have that

$$\forall i \; \sum_{j \neq i} v_j(\chi(v_{-i})) \geq \sum_{j \neq i} v_j(\chi(v)).$$

Thus the result follows directly. □

Bad news

Theorem

No dominant strategy incentive-compatible mechanism is always both efficient and weakly budget balanced, even if agents are restricted to the simple exchange setting.

Theorem

No Bayes-Nash incentive-compatible mechanism is always simultaneously efficient, weakly budget balanced and ex-interim individual rational, even if agents are restricted to quasilinear utility functions.