Risk Attitudes; Groves Mechanism

CPSC 532A Lecture 18

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Lecture Overview

1. Recap
2. Risk Attitudes
3. Quasilinear Mechanisms
4. Properties
5. The Groves Mechanism
Revelation Principle

- It turns out that truthfulness can always be achieved!
- Consider an arbitrary, non-truthful mechanism (e.g., may be indirect)
- Recall that a mechanism defines a game, and consider an equilibrium \( s = (s_1, \ldots, s_n) \)
We can construct a new direct mechanism, as shown above.

This mechanism is truthful by exactly the same argument that \( s \) was an equilibrium in the original mechanism.

“The agents don’t have to lie, because the mechanism already lies for them.”
Impossibility Result

Theorem (Gibbard-Satterthwaite)

Consider any social choice function $C$ of $N$ and $O$. If:

1. $|O| \geq 3$ (there are at least three outcomes);
2. $C$ is onto; that is, for every $o \in O$ there is a preference vector $\succ$ such that $C(\succ) = o$ (this property is sometimes also called citizen sovereignty); and
3. $C$ is dominant-strategy truthful,

then $C$ is dictatorial.
Quasilinear Utility

Definition (Quasilinear preferences)

Agents have quasilinear preferences in an $n$-player Bayesian game when the set of outcomes is $O = X \times \mathbb{R}^n$ for a finite set $X$, and the utility of an agent $i$ with type $\theta_i$ is given by

$$u_i(o, \theta_i) = u_i(x, \theta_i) - f_i(p_i),$$

where $o = (x, p_i)$ is an element of $O$, $u_i(x, \theta_i)$ is an arbitrary function and $f_i : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly monotonically increasing function.
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Fun game

- Look at your piece of paper: it contains an integer $x$.
- Go around the room offering everyone the following gamble:
  - they pay you $x$
  - you flip a coin:
    - heads: they win and get paid $2x$
    - tails: they lose and get nothing.
- Players can accept the gamble or decline.
  - Answer honestly (imagining the amounts of money are real)
  - play the gamble to see what would have happened.
- Keep track of:
  - Your own “bank balance” from others’ gambles you accepted.
  - The number of people who accepted your offer.
Risk Attitudes

- How much is $1 worth?
- What are the units in which this question should be answered?
Risk Attitudes

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Risk Attitudes

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    - Utils (units of utility)
  - Different amounts depending on the amount of money you already have
Risk Attitudes

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- How much is a gamble with an expected value of $1 worth?
How much is $1 worth?
- What are the units in which this question should be answered? Utils (units of utility)
- Different amounts depending on the amount of money you already have

How much is a gamble with an expected value of $1 worth?
- Possibly different amounts, depending on how risky it is
Risk Neutrality

(a) Risk neutrality

(b) Risk neutrality: fair lottery
Risk Aversion

(c) Risk aversion

(d) Risk aversion: fair lottery
Risk Seeking

(e) Risk seeking

(f) Risk seeking: fair lottery
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Quasilinear Mechanisms

**Definition (Quasilinear mechanism)**

A mechanism in the quasilinear setting (over a set of agents \( N \) and a set of outcomes \( O = X \times \mathbb{R}^n \)) is a triple \((A, \chi, p)\), where

- \( A = A_1 \times \cdots \times A_n \), where \( A_i \) is the set of actions available to agent \( i \in N \),
- \( \chi : A \rightarrow \Pi(X) \) maps each action profile to a distribution over choices, and
- \( p : A \rightarrow \mathbb{R}^n \) maps each action profile to a payment for each agent.
Quasilinear Mechanisms

Definition (Direct quasilinear mechanism)

A direct quasilinear mechanism (over a set of agents $N$ and a set of outcomes $O = X \times \mathbb{R}^n$) is a pair $(\chi, p)$. It defines a standard mechanism in the quasilinear setting, where for each $i$, $A_i = \Theta_i$.

- An agent's valuation for choice $x \in X$: $v_i(x) = u_i(x, \theta)$
  - the maximum amount $i$ would be willing to pay to get $x$
  - in fact, $i$ would be indifferent between keeping the money and getting $x$
- Equivalent definition: mechanisms that ask agents $i$ to declare $v_i(x)$ for each $x \in X$
- Define $\hat{v}_i$ as the valuation that agent $i$ declares to such a direct mechanism
  - may be different from his true valuation $v_i$
- Also define the tuples $\hat{v}, \hat{v}_{-i}$
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Truthfulness

**Definition (Truthfulness)**

A mechanism is *truthful* if $\forall i \forall v_i$, agent $i$’s equilibrium strategy is to adopt the strategy $\hat{v}_i = v_i$.

- Our definition before, adapted for the quasilinear setting
Efficiency

Definition (Efficiency)

A mechanism is efficient if it selects a choice \( x \) such that
\[
\forall i \forall v_i \forall x', \sum_i v_i(x) \geq \sum_i v_i(x').
\]

- An efficient mechanism selects the choice that maximizes the sum of agents’ utilities, disregarding monetary payments.
- Called economic efficiency to distinguish from other (e.g., computational) notions
- Also called social-welfare maximization
- Note: defined in terms of true (not declared) valuations, not declared valuations.
Budget Balance

**Definition (Budget balance)**

A mechanism is **budget balanced** when \( \forall \hat{v}, \sum_i p_i(\hat{v}) = 0 \).

- regardless of the agents’ types, the mechanism collects and disburses the same amount of money from and to the agents
- relaxed version: **weak budget balance**: \( \forall \hat{v} \sum_i p_i(\hat{v}) \geq 0 \)
  - the mechanism never takes a loss, but it may make a profit
- Budget balance can be required to hold *ex ante*: \( \mathbb{E}_v \sum_i p_i(v) = 0 \)
  - the mechanism must break even or make a profit only on expectation
Individual-Rationality

Definition (*Ex-interim* individual rationality)

A mechanism is *ex-interim individual rational* when

\[ \forall i \forall v, \mathbb{E}_{v_i \mid v_i} v_i(\chi(s_i(v_i), s_{-i}(v_{-i}))) - p_i(s_i(v_i), s_{-i}(v_{-i})) \geq 0, \]

where \( s \) is the equilibrium strategy profile.

- no agent loses by participating in the mechanism.
- *ex-interim* because it holds for every possible valuation for agent \( i \), but averages over the possible valuations of the other agents.

Definition (*Ex-post* individual rationality)

A mechanism is *ex-post individual rational* when

\[ \forall i \forall v, v_i(\chi(s(v))) - p_i(s(v)) \geq 0, \]

where \( s \) is the equilibrium strategy profile.
Tractability

**Definition (Tractability)**

A mechanism is **tractable** when $\forall \hat{v}, \chi(\hat{v})$ and $p(\hat{v})$ can be computed in polynomial time.

- The mechanism is computationally feasible.
Revenue Maximization

**Definition (Revenue maximization)**

A mechanism is *revenue maximizing* when, among the set of functions $\chi$ and $p$ which satisfy the other constraints, the mechanism selects the $\chi$ and $p$ which maximize $\mathbb{E}_\theta \sum_i p_i(s(\theta))$, where $s(\theta)$ denotes the agents’ equilibrium strategy.

- The mechanism designer can choose among mechanisms that satisfy the desired constraints by adding an objective function such as revenue maximization.
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A positive result

- Recall that in the quasilinear utility setting, a mechanism can be defined as a **choice rule** and a **payment rule**.
- The **Groves mechanism** is a mechanism that satisfies:
  - dominant strategy (truthfulness)
  - efficiency
- In general it’s not:
  - budget balanced
  - individual-rational

...though we’ll see later that there’s some hope for recovering these properties.
The Groves mechanism is a direct quasilinear mechanism $(\mathbb{R}^{|X|n}, \chi, p)$, where

\[ \chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x) \]

\[ p_i(\hat{v}) = h_i(\hat{v}_{-i}) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \]
The choice rule should not come as a surprise (why not?)
The choice rule should not come as a surprise (why not?) because the mechanism is both truthful and efficient: these properties entail the given choice rule.

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\chi(\hat{v}) = \arg \max_x \sum_i \hat{v}_i(x)
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The choice rule should not come as a surprise (why not?) because the mechanism is both truthful and efficient: these properties entail the given choice rule.

So what’s going on with the payment rule?

- the agent $i$ must pay some amount $h_i(\hat{v}_-i)$ that doesn't depend on his own declared valuation
- the agent $i$ is paid $\sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$, the sum of the others’ valuations for the chosen outcome
Groves Truthfulness

**Theorem**

*Truth telling is a dominant strategy under the Groves mechanism.*

Consider a situation where every agent $j$ other than $i$ follows some arbitrary strategy $\hat{v}_j$. Consider agent $i$’s problem of choosing the best strategy $\hat{v}_i$. As a shorthand, we will write $\hat{v} = (\hat{v}_{-i}, \hat{v}_i)$. The best strategy for $i$ is one that solves

$$\max_{\hat{v}_i} \left( v_i(\chi(\hat{v})) - p(\hat{v}) \right)$$

Substituting in the payment function from the Groves mechanism, we have

$$\max_{\hat{v}_i} \left( v_i(\chi(\hat{v})) - h_i(\hat{v}_{-i}) + \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \right)$$

Since $h_i(\hat{v}_{-i})$ does not depend on $\hat{v}_i$, it is sufficient to solve

$$\max_{\hat{v}_i} \left( v_i(\chi(\hat{v})) + \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \right).$$
Groves Truthfulness

\[
\max_{\hat{v}_i} \left( v_i(\chi(\hat{v})) + \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \right).
\]

The only way the declaration \( \hat{v}_i \) influences this maximization is through the choice of \( x \). If possible, \( i \) would like to pick a declaration \( \hat{v}_i \) that will lead the mechanism to pick an \( x \in X \) which solves

\[
\max_x \left( v_i(x) + \sum_{j \neq i} \hat{v}_j(x) \right).
\]

Under the Groves mechanism,

\[
\chi(\hat{v}) = \arg \max_x \left( \sum_i \hat{v}_i(x) \right) = \arg \max_x \left( \hat{v}_i(x) + \sum_{j \neq i} \hat{v}_j(x) \right).
\]

The Groves mechanism will choose \( x \) in a way that solves the maximization problem in Equation (1) when \( i \) declares \( \hat{v}_i = v_i \). Because this argument does not depend in any way on the declarations of the other agents, truth-telling is a dominant strategy for agent \( i \).
Proof intuition

- externalities are internalized
  - agents may be able to change the outcome to another one that they prefer, by changing their declaration
  - however, their utility doesn’t just depend on the outcome; it also depends on their payment
  - since they get paid the (reported) utility of all the other agents under the chosen allocation, they now have an interest in maximizing everyone’s utility rather than just their own

- in general, DS truthful mechanisms have the property that an agent’s payment doesn’t depend on the amount of his declaration, but only on the other agents’ declarations
  - the agent’s declaration is used only to choose the outcome, and to set other agents’ payments

- we’ll see later that Groves is the only truthful DS mechanism that is efficient