Risk Attitudes; Groves Mechanism

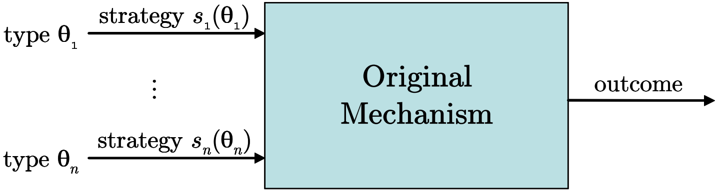
CPSC 532A Lecture 18

November 14, 2006

Lecture Overview

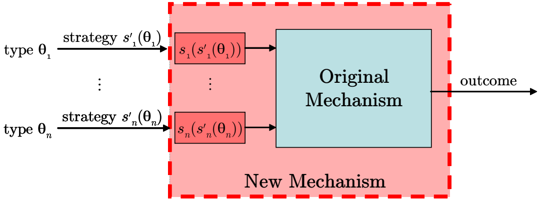
1. Recap
2. Risk Attitudes
3. Quasilinear Mechanisms
4. Properties
5. The Groves Mechanism

Revelation Principle



- It turns out that truthfulness can always be achieved!
- Consider an arbitrary, non-truthful mechanism (e.g., may be indirect)
- Recall that a mechanism defines a game, and consider an equilibrium $s = (s_1, \ldots, s_n)$

Revelation Principle



- We can construct a new direct mechanism, as shown above
- This mechanism is truthful by exactly the same argument that s was an equilibrium in the original mechanism
- "The agents don't have to lie, because the mechanism already lies for them."

Impossibility Result

Theorem (Gibbard-Satterthwaite)

Consider any social choice function C of N and O. If:

1. $|O| \geq 3$ *(there are at least three outcomes);*
2. C *is onto; that is, for every* $o \in O$ *there is a preference vector* $\succ$ *such that* $C(\succ) = o$ *(this property is sometimes also called citizen sovereignty); and*
3. C *is dominant-strategy truthful,*

then C *is dictatorial.*

Quasilinear Utility

Definition (Quasilinear preferences)

Agents have quasilinear preferences in an n-player Bayesian game when the set of outcomes is $O = X \times \mathbb{R}^n$ for a finite set X, and the utility of an agent i with type θ_i is given by $u_i(o, \theta_i) = u_i(x, \theta_i) - f_i(p_i)$, where $o = (x, p_i)$ is an element of O, $u_i(x, \theta_i)$ is an arbitrary function and $f_i : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly monotonically increasing function.

Lecture Overview

Fun game

- Look at your piece of paper: it contains an integer x
- Go around the room offering everyone the following gamble:
  - they pay you x
  - you flip a coin:
    - heads: they win and get paid $2x$
    - tails: they lose and get nothing.
  - Players can accept the gamble or decline.
    - Answer honestly (imagining the amounts of money are real)
    - play the gamble to see what would have happened.
  - Keep track of:
    - Your own "bank balance" from others' gambles you accepted.
    - The number of people who accepted your offer.

Risk Attitudes

- How much is \$1 worth?
  - What are the units in which this question should be answered?

Risk Attitudes

- How much is \$1 worth?
  - What are the units in which this question should be answered? Utils (units of utility)

Risk Attitudes

- How much is $1 worth?
  - What are the units in which this question should be answered? Utils (units of utility)
  - Different amounts depending on the amount of money you already have

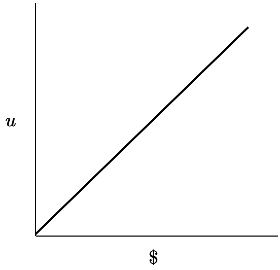
Risk Attitudes

- How much is \$1 worth?
  - What are the units in which this question should be answered? Utils (units of utility)
  - Different amounts depending on the amount of money you already have
- How much is a gamble with an expected value of \$1 worth?

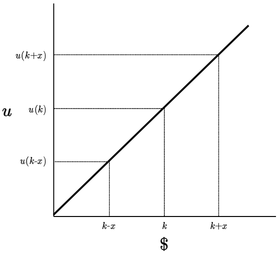
Risk Attitudes

- How much is \$1 worth?
  - What are the units in which this question should be answered? Utils (units of utility)
  - Different amounts depending on the amount of money you already have
- How much is a gamble with an expected value of \$1 worth?
  - Possibly different amounts, depending on how risky it is

Risk Neutrality

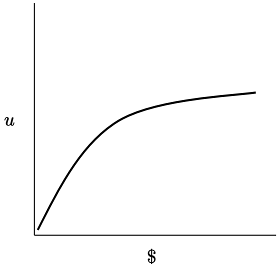


(a) Risk neutrality

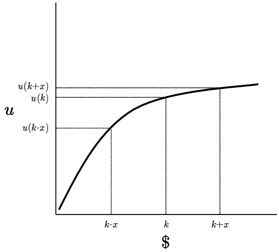


(b) Risk neutrality: fair lottery

Risk Aversion

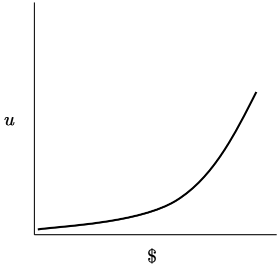


(c) Risk aversion

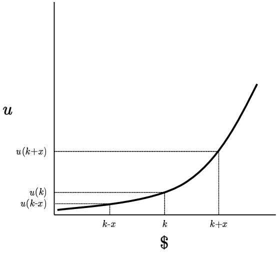


(d) Risk aversion: fair lottery

Risk Seeking



(e) Risk seeking



(f) Risk seeking: fair lottery

Lecture Overview

1. Recap
2. Risk Attitudes
3. Quasilinear Mechanisms
4. Properties
5. The Groves Mechanism

Quasilinear Mechanisms

Definition (Quasilinear mechanism)

A mechanism in the quasilinear setting (over a set of agents N and a set of outcomes $O = X \times \mathbb{R}^n$) is a triple (A, χ, p), where

- $A = A_1 \times \cdots \times A_n$, where A_i is the set of actions available to agent $i \in N$,
- $\chi : A \to \Pi(X)$ maps each action profile to a distribution over choices, and
- $p : A \to \mathbb{R}^n$ maps each action profile to a payment for each agent.

Quasilinear Mechanisms

Definition (Direct quasilinear mechanism)

A *direct quasilinear mechanism* (over a set of agents N and a set of outcomes $O = X \times \mathbb{R}^n$) is a pair (χ, p). It defines a standard mechanism in the quasilinear setting, where for each i, $A_i = \Theta_i$.

- An agent's valuation for choice $x \in X$: $v_i(x) = u_i(x, \theta)$
  - the maximum amount i would be willing to pay to get x
  - in fact, i would be indifferent between keeping the money and getting x
- Equivalent definition: mechanisms that ask agents i to declare $v_i(x)$ for each $x \in X$
- Define $\hat{v}_i$ as the valuation that agent i declares to such a direct mechanism
  - may be different from his true valuation v_i
- Also define the tuples $\hat{v}$, $\hat{v}_{-i}$

Lecture Overview

1. Recap
2. Risk Attitudes
3. Quasilinear Mechanisms
4. Properties
5. The Groves Mechanism

Truthfulness

Definition (Truthfulness)

A mechanism is *truthful* if $\forall i \forall v_i$, agent i's equilibrium strategy is to adopt the strategy $\hat{v}_i = v_i$.

- Our definition before, adapted for the quasilinear setting

Efficiency

Definition (Efficiency)

A mechanism is efficient if it selects a choice x such that $\forall i \forall v_i \forall x', \sum_i v_i(x) \geq \sum_i v_i(x')$.

- An efficient mechanism selects the choice that maximizes the sum of agents' utilities, disregarding monetary payments.
- Called economic efficiency to distinguish from other (e.g., computational) notions
- Also called social-welfare maximization
- Note: defined in terms of true (not declared) valuations, not declared valuations.

Budget Balance

Definition (Budget balance)

A mechanism is budget balanced when $\forall \hat{v},\ \sum_i p_i(\hat{v}) = 0$.

- regardless of the agents' types, the mechanism collects and disburses the same amount of money from and to the agents
- relaxed version: weak budget balance: $\forall \hat{v} \sum_i p_i(\hat{v}) \geq 0$
  - the mechanism never takes a loss, but it may make a profit
- Budget balance can be required to hold *ex ante*: $\mathbb{E}_v \sum_i p_i(v) = 0$
  - the mechanism must break even or make a profit only on expectation

Individual-Rationality

Definition (*Ex-interim* individual rationality)

A mechanism is ex-interim individual rational when
$\forall i \forall v_i,\ \mathbb{E}_{v_{-i}|v_i} v_i(\chi(s_i(v_i), s_{-i}(v_{-i}))) - p_i(s_i(v_i), s_{-i}(v_{-i})) \geq 0$,
where s is the equilibrium strategy profile.

- no agent loses by participating in the mechanism.
- *ex-interim* because it holds for *every* possible valuation for agent i, but averages over the possible valuations of the other agents.

Definition (*Ex-post* individual rationality)

A mechanism is ex-post individual rational when
$\forall i \forall v,\ v_i(\chi(s(v))) - p_i(s(v)) \geq 0$, where s is the equilibrium strategy profile.

Tractability

Definition (Tractability)

A mechanism is tractable when $\forall \hat{v}$, $\chi(\hat{v})$ and $p(\hat{v})$ can be computed in polynomial time.

- The mechanism is computationally feasible.

Revenue Maximization

Definition (Revenue maximization)

A mechanism is *revenue maximizing* when, among the set of functions χ and p which satisfy the other constraints, the mechanism selects the χ and p which maximize $\mathbb{E}_\theta \sum_i p_i(s(\theta))$, where $s(\theta)$ denotes the agents' equilibrium strategy.

- The mechanism designer can choose among mechanisms that satisfy the desired constraints by adding an objective function such as revenue maximization.

Lecture Overview

A positive result

- Recall that in the quasilinear utility setting, a mechanism can be defined as a choice rule and a payment rule.
- The Groves mechanism is a mechanism that satisfies:
  - dominant strategy (truthfulness)
  - efficiency
- In general it's not:
  - budget balanced
  - individual-rational

  ...though we'll see later that there's some hope for recovering these properties.

The Groves Mechanism

Definition (Groves mechanism)

The Groves mechanism is a direct quasilinear mechanism $(\mathbb{R}^{|X|n}, \chi, p)$, where

$$\chi(\hat{v}) = \arg\max_x \sum_i \hat{v}_i(x)$$

$$p_i(\hat{v}) = h_i(\hat{v}_{-i}) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

The Groves Mechanism

$$\chi(\hat{v}) = \arg\max_x \sum_i \hat{v}_i(x)$$

$$p_i(\hat{v}) = h_i(\hat{v}_{-i}) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

- The choice rule should not come as a surprise (why not?)

The Groves Mechanism

$$\chi(\hat{v}) = \arg\max_x \sum_i \hat{v}_i(x)$$

$$p_i(\hat{v}) = h_i(\hat{v}_{-i}) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

- The choice rule should not come as a surprise (why not?) because the mechanism is both truthful and efficient: these properties entail the given choice rule.

The Groves Mechanism

$$\chi(\hat{v}) = \arg\max_x \sum_i \hat{v}_i(x)$$

$$p_i(\hat{v}) = h_i(\hat{v}_{-i}) - \sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$$

- The choice rule should not come as a surprise (why not?) because the mechanism is both truthful and efficient: these properties entail the given choice rule.
- So what's going on with the payment rule?
  - the agent i must pay some amount $h_i(\hat{v}_{-i})$ that doesn't depend on his own declared valuation
  - the agent i is paid $\sum_{j \neq i} \hat{v}_j(\chi(\hat{v}))$, the sum of the others' valuations for the chosen outcome

Groves Truthfulness

Theorem

Truth telling is a dominant strategy under the Groves mechanism.

Consider a situation where every agent j other than i follows some arbitrary strategy $\hat{v}_j$. Consider agent i's problem of choosing the best strategy $\hat{v}_i$. As a shorthand, we will write $\hat{v} = (\hat{v}_{-i}, \hat{v}_i)$. The best strategy for i is one that solves

$$\max_{\hat{v}_i} \left(v_i(\chi(\hat{v})) - p(\hat{v}) \right)$$

Substituting in the payment function from the Groves mechanism, we have

$$\max_{\hat{v}_i} \left(v_i(\chi(\hat{v})) - h_i(\hat{v}_{-i}) + \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \right)$$

Since $h_i(\hat{v}_{-i})$ does not depend on $\hat{v}_i$, it is sufficient to solve

$$\max_{\hat{v}_i} \left(v_i(\chi(\hat{v})) + \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \right).$$

Groves Truthfulness

$$\max_{\hat{v}_i} \left(v_i(\chi(\hat{v})) + \sum_{j \neq i} \hat{v}_j(\chi(\hat{v})) \right).$$

The only way the declaration $\hat{v}_i$ influences this maximization is through the choice of x. If possible, i would like to pick a declaration $\hat{v}_i$ that will lead the mechanism to pick an $x \in X$ which solves

$$\max_{x} \left(v_i(x) + \sum_{j \neq i} \hat{v}_j(x) \right). \quad (1)$$

Under the Groves mechanism,

$$\chi(\hat{v}) = \arg\max_{x} \left(\sum_{i} \hat{v}_i(x) \right) = \arg\max_{x} \left(\hat{v}_i(x) + \sum_{j \neq i} \hat{v}_j(x) \right).$$

The Groves mechanism will choose x in a way that solves the maximization problem in Equation (**??**) when i declares $\hat{v}_i = v_i$. Because this argument does not depend in any way on the declarations of the other agents, truth-telling is a dominant strategy for agent i.

Proof intuition

- externalities are internalized
  - agents may be able to change the outcome to another one that they prefer, by changing their declaration
  - however, their utility doesn't just depend on the outcomeit also depends on their payment
  - since they get paid the (reported) utility of all the other agents under the chosen allocation, they now have an interest in maximizing everyone's utility rather than just their own
- in general, DS truthful mechanisms have the property that an agent's payment doesn't depend on the amount of his declaration, but only on the other agents' declarations
  - the agent's declaration is used only to choose the outcome, and to set other agents' payments
- we'll see later that Groves is the only truthful DS mechanism that is efficient