Social Choice, Arrow's Theorem

CPSC 532A Lecture 15

October 26, 2006

Lecture Overview

Recap

Voting Paradoxes

Fun Game

Properties

Arrow's Theorem

Ex-post expected utility

Definition (*Ex-post* expected utility)

Agent i's *ex-post* expected utility in a Bayesian game (N, A, Θ, p, u), where the agents' strategies are given by s and the agent' types are given by θ, is defined as

$$EU_i(s, \theta) = \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j | \theta_j) \right) u_i(a, \theta).$$

- The only uncertainty here concerns the other agents' mixed strategies, since i knows everyone's type.

Ex-interim expected utility

Definition (*Ex-interim* expected utility)

Agent i's *ex-interim* expected utility in a Bayesian game (N, A, Θ, p, u), where i's type is θ_i and where the agents' strategies are given by the mixed strategy profile s, is defined as

$$EU_i(s|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a, \theta_{-i}, \theta_i).$$

- i must consider every θ_{-i} and every a in order to evaluate $u_i(a, \theta_i, \theta_{-i})$.
- i must weight this utility value by:
  - the probability that a would be realized given all players' mixed strategies and types;
  - the probability that the other players' types would be θ_{-i} given that his own type is θ_i.

Ex-ante expected utility

Definition (*Ex-ante* expected utility)

Agent i's *ex-ante* expected utility in a Bayesian game (N, A, Θ, p, u), where the agents' strategies are given by the mixed strategy profile s, is defined as

$$EU_i(s) = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s|\theta_i)$$

or equivalently as

$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a, \theta).$$

Nash equilibrium

Definition (Bayes-Nash equilibrium)

A Bayes-Nash equilibrium is a mixed strategy profile s that satisfies $\forall i \;\; s_i \in BR_i(s_{-i})$.

Definition (*ex-post* Bayes-Nash equilibrium)

A *ex-post* Bayes-Nash equilibrium is a mixed strategy profile s that satisfies $\forall \theta,\ \forall i,\ s_i \in \arg\max_{s'_i \in S_i} EU_i(s'_i, s_{-i}, \theta)$.

- almost a dominant strategy, but not quite

Social Choice

Definition (Social choice function)

Assume a set of agents $N = \{1, 2, \ldots, n\}$, and a set of outcomes (or alternatives, or candidates) O. Let L be the set of strict total orders on O. A social choice function (over N and O) is a function $C : L^n \rightarrow O$.

Definition (Social welfare function)

Let N, O, L be as above. A social welfare function (over N and O) is a function $W : L^n \rightarrow L^-$, where L^- is the set of weak total orderings (that is, total preorders) on O.

Some Voting Schemes

- Plurality
  - pick the outcome which is preferred by the most people
- Plurality with elimination ("instant runoff")
  - everyone selects their favorite outcome
  - the outcome with the fewest votes is eliminated
  - repeat until one outcome remains
- Borda
  - assign each outcome a number.
  - The most preferred outcome gets a score of $n - 1$, the next most preferred gets $n - 2$, down to the n^{th} outcome which gets 0.
  - Then sum the numbers for each outcome, and choose the one that has the highest score
- Pairwise elimination
  - in advance, decide a schedule for the order in which pairs will be compared.
  - given two outcomes, have everyone determine the one that they prefer

Condorcet Condition

- If there is a candidate who is preferred to every other candidate in pairwise runoffs, that candidate should be the winner
- While the Condorcet condition is considered an important property for a voting system to satisfy, there is not always a Condorcet winner
- sometimes, there's a cycle where A defeats B, B defeats C, and C defeats A in their pairwise runoffs

Lecture Overview

Sensitivity to Losing Candidate

35 agents: $A \succ C \succ B$
33 agents: $B \succ A \succ C$
32 agents: $C \succ B \succ A$

- What candidate wins under plurality voting?

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Sensitivity to Losing Candidate

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- What candidate wins under plurality voting? A
- What candidate wins under Borda voting?

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- What candidate wins under plurality voting? A
- What candidate wins under Borda voting? A
- Now consider dropping C. Now what happens under both Borda and plurality?

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- What candidate wins under plurality voting? A
- What candidate wins under Borda voting? A
- Now consider dropping C. Now what happens under both Borda and plurality? B wins.

Sensitivity to Agenda Setter

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- Who wins pairwise elimination, with the ordering A, B, C?

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- Who wins with the ordering B, C, A?

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- Who wins with the ordering A, C, B? B
- Who wins with the ordering B, C, A? A

Another Pairwise Elimination Problem

1 agent: $B \succ D \succ C \succ A$
1 agent: $A \succ B \succ D \succ C$
1 agent: $C \succ A \succ B \succ D$

- Who wins under pairwise elimination with the ordering A, B, C, D?

Another Pairwise Elimination Problem

1 agent: $B \succ D \succ C \succ A$
1 agent: $A \succ B \succ D \succ C$
1 agent: $C \succ A \succ B \succ D$

- Who wins under pairwise elimination with the ordering A, B, C, D? D.

Another Pairwise Elimination Problem

1 agent: $B \succ D \succ C \succ A$
1 agent: $A \succ B \succ D \succ C$
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- Who wins under pairwise elimination with the ordering A, B, C, D? D.
- What is the problem with this?

Another Pairwise Elimination Problem

$$\begin{aligned} \text{1 agent:} \quad & B \succ D \succ C \succ A \\ \text{1 agent:} \quad & A \succ B \succ D \succ C \\ \text{1 agent:} \quad & C \succ A \succ B \succ D \end{aligned}$$

- Who wins under pairwise elimination with the ordering A, B, C, D? D.
- What is the problem with this?
  - *all* of the agents prefer B to D—the selected candidate is Pareto-dominated!

Lecture Overview

Fun Game

- Imagine that there was an opportunity to take a one-week class trip at the end of term, to one of the following destinations:
  - (O) Orlando, FL
  - (L) London, England
  - (M) Moscow, Russia
  - (B) Beijing, China
- Construct your preference ordering

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- Vote (truthfully) using each of the following schemes:
  - plurality (raise hands)

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  - plurality (raise hands)
  - plurality with elimination (raise hands)

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- Construct your preference ordering
- Vote (truthfully) using each of the following schemes:
  - plurality (raise hands)
  - plurality with elimination (raise hands)
  - Borda (volunteer to tabulate)
  - pairwise elimination (raise hands, I'll pick a schedule)

Lecture Overview

Notation

- N is the set of agents
- O is a finite set of outcomes with $|O| \geq 3$
- L the set of all possible preference orderings over O.
- $\succ$ is an element of the set L^n (a preference ordering for every agent; the input to our social welfare function)
- $\succ_W$ is the preference ordering selected by the social welfare function W.
  - When the input to W is ambiguous we write it in the subscript; thus, the social order selected by W given the input $\succ'$ is denoted as $\succ_{W(\succ')}$.

Pareto Efficiency

Definition (Pareto Efficiency (PE))

W is Pareto efficient if for any $o_1, o_2 \in O$, $\forall i\, o_1 \succ_i o_2$ implies that $o_1 \succ_W o_2$.

- when all agents agree on the ordering of two outcomes, the social welfare function must select that ordering.

Independence of Irrelevant Alternatives

Definition (Independence of Irrelevant Alternatives (IIA))

W is independent of irrelevant alternatives if, for any $o_1, o_2 \in O$ and any two preference profiles $\succ', \succ'' \in L^n$, $\forall i\,(o_1 \succ'_i o_2 \leftrightarrow o_1 \succ''_i o_2)$ implies that $o_1 \succ_{W(\succ')} o_2 \Leftrightarrow o_1 \succ_{W(\succ'')} o_2$.

- the selected ordering between two outcomes should depend only on the relative orderings they are given by the agents.

Nondictatorship

Definition (Non-dictatorship)

W does not have a dictator if $\neg\exists i\, \forall o_1, o_2 (o_1 \succ_i o_2 \Rightarrow o_1 \succ_W o_2)$.

- there does not exist a single agent whose preferences always determine the social ordering.
- We say that W is *dictatorial* if it fails to satisfy this property.

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Theorem (Arrow, 1951)

Any social welfare function W that is Pareto efficient and independent of irrelevant alternatives is dictatorial.

We will assume that W is both PE and IIA, and show that W must be dictatorial. The argument proceeds in four steps.

Step 1

If every voter puts an outcome b at either the very top or the very bottom of his preference list, b must be at either the very top or very bottom of $\succ_W$ as well.

Consider an arbitrary preference profile $\succ$ in which every voter ranks some $b \in O$ at either the very bottom or very top, and assume for contradiction that the above claim is not true. Then, there must exist some pair of distinct outcomes $a, c \in O$ for which $a \succ_W b$ and $b \succ_W c$.

Step 1

If every voter puts an outcome b at either the very top or the very bottom of his preference list, b must be at either the very top or very bottom of $\succ_W$ as well.

Now let's modify $\succ$ so that every voter moves c just above a in his preference ranking, and otherwise leaves the ranking unchanged; let's call this new preference ordering $\succ'$. We know from IIA that for $a \succ_W b$ or $b \succ_W c$ to change, the pairwise relationship between a and b and/or the pairwise relationship between b and c would have to change. However, since b occupies an extremal position for all voters, c can be moved above a without changing either of these pairwise relationships. Thus in profile $\succ'$ it is also the case that $a \succ_W b$ and $b \succ_W c$. From this fact and from transitivity, we have that $a \succ_W c$. However, in $\succ'$ every voter ranks c above a and so PE requires that $c \succ_W a$. We have a contradiction.