Social Choice, Arrow’s Theorem

CPSC 532A Lecture 15

October 26, 2006
Recap

Voting Paradoxes

Fun Game

Properties

Arrow’s Theorem
**Ex-post expected utility**

**Definition (Ex-post expected utility)**

Agent $i$’s *ex-post expected utility* in a Bayesian game $(N, A, \Theta, p, u)$, where the agents’ strategies are given by $s$ and the agent’s types are given by $\theta$, is defined as

$$EU_i(s, \theta) = \sum_{a \in A} \left( \prod_{j \in N} s_j(a_j | \theta_j) \right) u_i(a, \theta).$$

- The only uncertainty here concerns the other agents’ mixed strategies, since $i$ knows everyone’s type.
**Ex-interim expected utility**

**Definition (Ex-interim expected utility)**

Agent $i$’s *ex-interim expected utility* in a Bayesian game $(N, A, \Theta, p, u)$, where $i$’s type is $\theta_i$ and where the agents’ strategies are given by the mixed strategy profile $s$, is defined as

$$EU_i(s|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}|\theta_i) \sum_{a \in A} \left( \prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a, \theta_{-i}, \theta_i).$$

- $i$ must consider every $\theta_{-i}$ and every $a$ in order to evaluate $u_i(a, \theta_i, \theta_{-i})$.
- $i$ must weight this utility value by:
  - the probability that $a$ would be realized given all players’ mixed strategies and types;
  - the probability that the other players’ types would be $\theta_{-i}$ given that his own type is $\theta_i$. 

Social Choice, Arrow’s Theorem

CPSC 532A Lecture 15, Slide 4
Definition (Ex-ante expected utility)

Agent $i$’s \textit{ex-ante expected utility} in a Bayesian game $(N, A, \Theta, p, u)$, where the agents’ strategies are given by the mixed strategy profile $s$, is defined as

\[
EU_i(s) = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s|\theta_i)
\]

or equivalently as

\[
EU_i(s) = \sum_{\theta \in \Theta} p(\theta) \sum_{a \in A} \left( \prod_{j \in N} s_j(a_j|\theta_j) \right) u_i(a, \theta).
\]
Nash equilibrium

Definition (Bayes-Nash equilibrium)

A Bayes-Nash equilibrium is a mixed strategy profile $s$ that satisfies
$\forall i \ s_i \in BR_i(s_{-i})$.

Definition (ex-post Bayes-Nash equilibrium)

A ex-post Bayes-Nash equilibrium is a mixed strategy profile $s$ that satisfies
$\forall \theta, \forall i, s_i \in \arg \max_{s_i' \in S_i} EU_i(s_i', s_{-i}, \theta)$.

► almost a dominant strategy, but not quite
Social Choice

Definition (Social choice function)
Assume a set of agents $N = \{1, 2, \ldots, n\}$, and a set of outcomes (or alternatives, or candidates) $O$. Let $L$ be the set of strict total orders on $O$. A social choice function (over $N$ and $O$) is a function $C : L^n \to O$.

Definition (Social welfare function)
Let $N, O, L$ be as above. A social welfare function (over $N$ and $O$) is a function $W : L^n \to L^-$, where $L^-$ is the set of weak total orderings (that is, total preorders) on $O$. 
Some Voting Schemes

- **Plurality**
  - pick the outcome which is preferred by the most people

- **Plurality with elimination ("instant runoff")**
  - everyone selects their favorite outcome
  - the outcome with the fewest votes is eliminated
  - repeat until one outcome remains

- **Borda**
  - assign each outcome a number.
  - The most preferred outcome gets a score of \( n - 1 \), the next most preferred gets \( n - 2 \), down to the \( n^{th} \) outcome which gets 0.
  - Then sum the numbers for each outcome, and choose the one that has the highest score

- **Pairwise elimination**
  - in advance, decide a schedule for the order in which pairs will be compared.
  - given two outcomes, have everyone determine the one that they prefer
Condorcet Condition

- If there is a candidate who is preferred to every other candidate in pairwise runoffs, that candidate should be the winner.
- While the Condorcet condition is considered an important property for a voting system to satisfy, there is not always a Condorcet winner.
- Sometimes, there's a cycle where A defeats B, B defeats C, and C defeats A in their pairwise runoffs.
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Lecture Overview
Sensitivity to Losing Candidate

35 agents:  $A \succ C \succ B$
33 agents:  $B \succ A \succ C$
32 agents:  $C \succ B \succ A$

▶ What candidate wins under plurality voting?
Sensitivity to Losing Candidate

35 agents: $A \succ C \succ B$
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What candidate wins under plurality voting? $A$
Sensitivity to Losing Candidate

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- What candidate wins under plurality voting? $A$
- What candidate wins under Borda voting?
Sensitivity to Losing Candidate

35 agents:  \( A \succ C \succ B \)
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- What candidate wins under plurality voting?  \( A \)
- What candidate wins under Borda voting?  \( A \)
Sensitivity to Losing Candidate

- 35 agents: $A \succ C \succ B$
- 33 agents: $B \succ A \succ C$
- 32 agents: $C \succ B \succ A$

- What candidate wins under plurality voting? $A$
- What candidate wins under Borda voting? $A$
- Now consider dropping $C$. Now what happens under both Borda and plurality?
Sensitivity to Losing Candidate

- What candidate wins under plurality voting? A
- What candidate wins under Borda voting? A
- Now consider dropping C. Now what happens under both Borda and plurality? B wins.
Sensitivity to Agenda Setter

35 agents: $A \succ C \succ B$
33 agents: $B \succ A \succ C$
32 agents: $C \succ B \succ A$

- Who wins pairwise elimination, with the ordering $A, B, C$?
Sensitivity to Agenda Setter

- 35 agents: $A \succ C \succ B$
- 33 agents: $B \succ A \succ C$
- 32 agents: $C \succ B \succ A$

—who wins pairwise elimination, with the ordering $A, B, C$? C
Sensitivity to Agenda Setter

35 agents: \( A \succ C \succ B \)
33 agents: \( B \succ A \succ C \)
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- Who wins pairwise elimination, with the ordering \( A, B, C \)? \( C \)
- Who wins with the ordering \( A, C, B \)?
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- Who wins pairwise elimination, with the ordering $A, B, C$? $C$
- Who wins with the ordering $A, C, B$? $B$
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- Who wins pairwise elimination, with the ordering \( A, B, C \)? \( C \)
- Who wins with the ordering \( A, C, B \)? \( B \)
- Who wins with the ordering \( B, C, A \)?
Sensitivity to Agenda Setter

- 35 agents: $A \succ C \succ B$
- 33 agents: $B \succ A \succ C$
- 32 agents: $C \succ B \succ A$

- Who wins pairwise elimination, with the ordering $A, B, C$? $C$
- Who wins with the ordering $A, C, B$? $B$
- Who wins with the ordering $B, C, A$? $A$
Another Pairwise Elimination Problem

1 agent: \[ B \succ D \succ C \succ A \]
1 agent: \[ A \succ B \succ D \succ C \]
1 agent: \[ C \succ A \succ B \succ D \]

Who wins under pairwise elimination with the ordering \( A, B, C, D \)?
Another Pairwise Elimination Problem

1 agent: \( B \succ D \succ C \succ A \)

1 agent: \( A \succ B \succ D \succ C \)

1 agent: \( C \succ A \succ B \succ D \)

Who wins under pairwise elimination with the ordering \( A, B, C, D \)? \( D \).
Another Pairwise Elimination Problem

1 agent: \( B \succ D \succ C \succ A \)
1 agent: \( A \succ B \succ D \succ C \)
1 agent: \( C \succ A \succ B \succ D \)

- Who wins under pairwise elimination with the ordering \( A, B, C, D \)? \( D \).
- What is the problem with this?
Another Pairwise Elimination Problem

1 agent: \( B \succ D \succ C \succ A \)
1 agent: \( A \succ B \succ D \succ C \)
1 agent: \( C \succ A \succ B \succ D \)

Who wins under pairwise elimination with the ordering \( A, B, C, D? \) \( D \).

What is the problem with this?

- all of the agents prefer \( B \) to \( D \)—the selected candidate is Pareto-dominated!
Lecture Overview

Recap

Voting Paradoxes

Fun Game

Properties

Arrow’s Theorem
Fun Game

- Imagine that there was an opportunity to take a one-week class trip at the end of term, to one of the following destinations:
  - (O) Orlando, FL
  - (L) London, England
  - (M) Moscow, Russia
  - (B) Beijing, China

- Construct your preference ordering
Fun Game

- Imagine that there was an opportunity to take a one-week class trip at the end of term, to one of the following destinations:
  - (O) Orlando, FL
  - (L) London, England
  - (M) Moscow, Russia
  - (B) Beijing, China
- Construct your preference ordering
- Vote (truthfully) using each of the following schemes:
  - plurality (raise hands)
Imagine that there was an opportunity to take a one-week class trip at the end of term, to one of the following destinations:

- (O) Orlando, FL
- (L) London, England
- (M) Moscow, Russia
- (B) Beijing, China

Construct your preference ordering

Vote (truthfully) using each of the following schemes:

- plurality (raise hands)
- plurality with elimination (raise hands)
Imagine that there was an opportunity to take a one-week class trip at the end of term, to one of the following destinations:

- (O) Orlando, FL
- (L) London, England
- (M) Moscow, Russia
- (B) Beijing, China

Construct your preference ordering

Vote (truthfully) using each of the following schemes:

- plurality (raise hands)
- plurality with elimination (raise hands)
- Borda (volunteer to tabulate)
Imagine that there was an opportunity to take a one-week class trip at the end of term, to one of the following destinations:

- (O) Orlando, FL
- (L) London, England
- (M) Moscow, Russia
- (B) Beijing, China

Construct your preference ordering

Vote (truthfully) using each of the following schemes:

- plurality (raise hands)
- plurality with elimination (raise hands)
- Borda (volunteer to tabulate)
- pairwise elimination (raise hands, I'll pick a schedule)
Lecture Overview

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Arrow’s Theorem
Notation

- $N$ is the set of agents
- $O$ is a finite set of outcomes with $|O| \geq 3$
- $L$ the set of all possible preference orderings over $O$.
- $\succ$ is an element of the set $L^n$ (a preference ordering for every agent; the input to our social welfare function)
- $\succ_W$ is the preference ordering selected by the social welfare function $W$.
  - When the input to $W$ is ambiguous we write it in the subscript; thus, the social order selected by $W$ given the input $\succ'$ is denoted as $\succ_W(\succ')$. 
Pareto Efficiency

Definition (Pareto Efficiency (PE))

$W$ is **Pareto efficient** if for any $o_1, o_2 \in O$, $\forall i o_1 \succ_i o_2$ implies that $o_1 \succ_W o_2$.

- when all agents agree on the ordering of two outcomes, the social welfare function must select that ordering.
Independence of Irrelevant Alternatives

Definition (Independence of Irrelevant Alternatives (IIA))

$W$ is **independent of irrelevant alternatives** if, for any $o_1, o_2 \in O$ and any two preference profiles $\succ', \succ'' \in L^n$, 

$$\forall i \left( o_1 \succ'_i o_2 \iff o_1 \succ''_i o_2 \right)$$

implies that

$$o_1 \succ_W (\succ') o_2 \iff o_1 \succ_W (\succ'') o_2.$$

- the selected ordering between two outcomes should depend only on the relative orderings they are given by the agents.
Definition (Non-dictatorship)

\( W \) does not have a dictator if \( \neg \exists i: \forall o_1, o_2 (o_1 \succ_i o_2 \Rightarrow o_1 \succ_W o_2) \).

- there does not exist a single agent whose preferences always determine the social ordering.
- We say that \( W \) is \textit{dictatorial} if it fails to satisfy this property.
Lecture Overview

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Arrow’s Theorem
Arrow’s Theorem

Theorem (Arrow, 1951)

Any social welfare function \( W \) that is Pareto efficient and independent of irrelevant alternatives is dictatorial.

We will assume that \( W \) is both PE and IIA, and show that \( W \) must be dictatorial. The argument proceeds in four steps.
Step 1

If every voter puts an outcome $b$ at either the very top or the very bottom of his preference list, $b$ must be at either the very top or very bottom of $\succ_W$ as well.

Consider an arbitrary preference profile $\succ$ in which every voter ranks some $b \in O$ at either the very bottom or very top, and assume for contradiction that the above claim is not true. Then, there must exist some pair of distinct outcomes $a, c \in O$ for which $a \succ_W b$ and $b \succ_W c$. 
Step 1

If every voter puts an outcome $b$ at either the very top or the very bottom of his preference list, $b$ must be at either the very top or very bottom of $\succ_W$ as well.

Now let’s modify $\succ$ so that every voter moves $c$ just above $a$ in his preference ranking, and otherwise leaves the ranking unchanged; let’s call this new preference ordering $\succ'$. We know from IIA that for $a \succ_W b$ or $b \succ_W c$ to change, the pairwise relationship between $a$ and $b$ and/or the pairwise relationship between $b$ and $c$ would have to change. However, since $b$ occupies an extremal position for all voters, $c$ can be moved above $a$ without changing either of these pairwise relationships. Thus in profile $\succ'$ it is also the case that $a \succ_W b$ and $b \succ_W c$. From this fact and from transitivity, we have that $a \succ_W c$. However, in $\succ'$ every voter ranks $c$ above $a$ and so PE requires that $c \succ_W a$. We have a contradiction.