Stochastic Games and Bayesian Games

CPSC 532A Lecture 13

October 24, 2006

Lecture Overview

Recap

Stochastic Games

Bayesian Games

Definitions

- Consider any n-player game $G = (N, (A_i), (u_i))$ and any payoff vector $r = (r_1, r_2, \ldots, r_n)$.
- Let $v_i = \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_{-i}, s_i)$.
  - the amount of utility i can get when $-i$ play a minmax strategy against him

Definition

A payoff profile r is enforceable if $r_i \geq v_i$.

Definition

A payoff profile r is feasible if there exist rational, non-negative values α_a such that for all i, we can express r_i as $\sum_{a \in A} \alpha u_i(a)$, with $\sum_{a \in A} \alpha_a = 1$.

- a payoff profile is feasible if it is a convex, rational combination of the outcomes in G.

Folk Theorem

Theorem (Folk Theorem)

Consider any n-player game G and any payoff vector $(r_1, r_2, \ldots, r_n)$.

1. *If r is the payoff in any Nash equilibrium of the infinitely repeated G with average rewards, then for each player i, r_i is enforceable.*
2. *If r is both feasible and enforceable, then r is the payoff in some Nash equilibrium of the infinitely repeated G with average rewards.*

Folk Theorem (Part 1)

Payoff in Nash $\rightarrow$ enforceable

Part 1: Suppose r is not enforceable, i.e. $r_i < v_i$ for some i. Then consider a deviation of this player i to $b_i(s_{-i}(h))$ for any history h of the repeated game, where b_i is any best-response action in the stage game and $s_{-i}(h)$ is the equilibrium strategy of other players given the current history h. By definition of a minmax strategy, player i will receive a payoff of at least v_i in every stage game if he adopts this strategy, and so i's average reward is also at least v_i. Thus i cannot receive the payoff $r_i < v_i$ in any Nash equilibrium.

Folk Theorem (Part 2)

Feasible and enforceable → Nash

Part 2: Since r is a feasible enforceable payoff profile, we can write it as $r_i = \sum_{a \in A} (\frac{\beta_a}{\gamma}) u_i(a)$, where β_a and γ are non-negative integers. (Recall that α_a were required to be rational. So we can take γ to be their common denominator.) Since the combination was convex, we have $\gamma = \sum_{a \in A} \beta_a$.

We're going to construct a strategy profile that will cycle through all outcomes $a \in A$ of G with cycles of length γ, each cycle repeating action a exactly β_a times. Let (a^t) be such a sequence of outcomes. Let's define a strategy s_i of player i to be a trigger version of playing (a^t): if nobody deviates, then s_i plays a_i^t in period t. However, if there was a period t' in which some player $j \neq i$ deviated, then s_i will play $(p_{-j})_i$, where (p_{-j}) is a solution to the minimization problem in the definition of v_j.

Folk Theorem (Part 2)

Feasible and enforceable $\rightarrow$ Nash

First observe that if everybody plays according to s_i, then, by construction, player i receives average payoff of r_i (look at averages over periods of length γ). Second, this strategy profile is a Nash equilibrium. Suppose everybody plays according to s_i, and player j deviates at some point. Then, forever after, player j will receive his $\min\max$ payoff $v_j \leq r_j$, rendering the deviation unprofitable.

Lecture Overview

Recap

Stochastic Games

Bayesian Games

Introduction

- What if we didn't always repeat back to the same stage game?
- A stochastic game is a generalization of repeated games
  - agents repeatedly play games from a set of normal-form games
  - the game played at any iteration depends on the previous game played and on the actions taken by all agents in that game
- A stochastic game is a generalized Markov decision process
  - there are multiple players
  - one reward function for each agent
  - the state transition function and reward functions depend on the action choices of both players

Formal Definition

Definition

A stochastic game is a tuple $(Q, N, A_1, \ldots, A_n, P, r_1, \ldots, r_n)$, where

- Q is a finite set of states,
- N is a finite set of n players,
- A_i is a finite set of actions available to player i. Let $A = A_1 \times \cdots \times A_n$ be the vector of all players' actions,
- $P : Q \times A \times Q \to [0, 1]$ is the transition probability function; let $P(q, a, \hat{q})$ be the probability of transitioning from state s to state $\hat{q}$ after joint action a,
- $r_i : Q \times A \to \mathbb{R}$ is a real-valued payoff function for player i.

Remarks

- This assumes strategy space same in all games; otherwise just more notation
- Again we can have average or discounted payoffs.
- Interesting special cases:
  - zero-sum stochastic game
  - single-controller stochastic game (in this latter, transitions (but not payoffs) depend on only one agent)

Strategies

- What is a pure strategy?

Strategies

- What is a pure strategy?
  - pick an action conditional on every possible history
  - of course, mixtures over these pure strategies are possible too!
- Some interesting restricted classes of strategies:
  - behavioral strategy: $s_i(h_t, a_{i_j})$ returns the probability of playing action a_{i_j} for history h_t.
    - the substantive assumption here is that mixing takes place at each history independently, not once at the beginning of the game
  - Markov strategy: s_i is a behavioral strategy in which $s_i(h_t, a_{i_j}) = s_i(h'_t, a_{i_j})$ if $q_t = q'_t$, where q_t and q'_t are the final states of h_t and h'_t, respectively.
    - for a given time t, the distribution over actions only depends on the current state
  - stationary strategy: s_i is a Markov strategy in which $s_i(h_{t_1}, a_{i_j}) = s_i(h'_{t_2}, a_{i_j})$ if $q_{t_1} = q'_{t_2}$, where q_{t_1} and q'_{t_2} are the final states of h_{t_1} and h'_{t_2}, respectively.
    - no dependence even on t

Equilibrium (discounted rewards)

- Markov perfect equilibrium:
  - a strategy profile consisting of only Markov strategies that is a Nash equilibrium regardless of the starting state
  - analogous to subgame-perfect equilibrium

Theorem

Every n-player, general sum, discounted reward stochastic game has a Markov perfect equilibrium.

Equilibrium (average rewards)

- Irreducible stochastic game:
  - every strategy profile gives rise to an irreducible Markov chain over the set of games
  - during the (infinite) execution of the stochastic game, each stage game is guaranteed to be played infinitely often—for any strategy profile
  - without this condition, limit of the mean payoffs may not be defined

Theorem

For every 2-player, general sum, average reward, irreducible stochastic game has a Nash equilibrium.

A folk theorem

Theorem

For every 2-player, general sum, irreducible stochastic game, and every feasible outcome with a payoff vector r *that provides to each player at least his maxmin value, there exists a Nash equilibrium with a payoff vector* r*. This is true for games with average rewards, as well as games with large enough discount factors (i.e. with players that are sufficiently patient).*

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Fun Game

- Choose a phone number none of your neighbours knows; consider it to be ABC-DEFG

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  - what is the role of uncertainty here?
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- Questions:
  - what is the role of uncertainty here?
  - can we model this uncertainty using an imperfect information extensive form game?
    - imperfect info means not knowing what node you’re in in the info set
    - here we’re not sure what game is being played (though if we allow a move by nature, we can do it)

Introduction

- So far, we've assumed that all players know what game is being played. Everyone knows:
  - the number of players
  - the actions available to each player
  - the payoff associated with each action vector
- Why is this true in imperfect information games?
- We'll assume:

1. All possible games have the same number of agents and the same strategy space for each agent; they differ only in their payoffs.
2. The beliefs of the different agents are posteriors, obtained by conditioning a common prior on individual private signals.

Definition 1: Information Sets

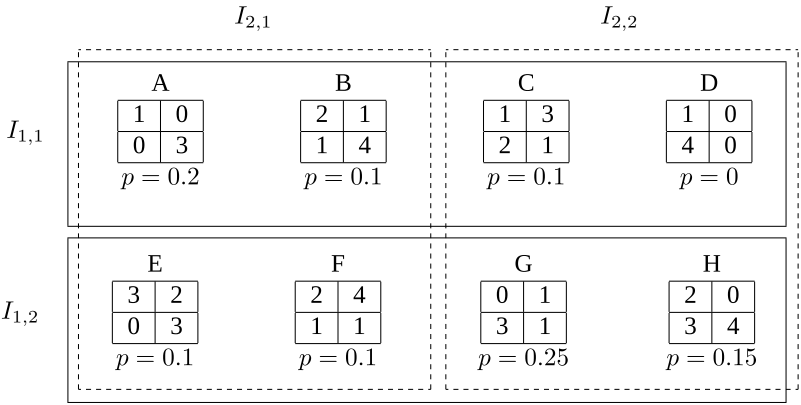
- Bayesian game: a set of games that differ only in their payoffs, a common prior defined over them, and a partition structure over the games for each agent.

Definition (Bayesian Game: Information Sets)

A Bayesian game is a tuple (N, G, P, I) where

- N is a set of agents,
- G is a set of games with N agents each such that if $g, g' \in G$ then for each agent $i \in N$ the strategy space in g is identical to the strategy space in g',
- $P \in \Pi(G)$ is a common prior over games, where $\Pi(G)$ is the set of all probability distributions over G, and
- $I = (I_1, ..., I_N)$ is a set of partitions of G, one for each agent.

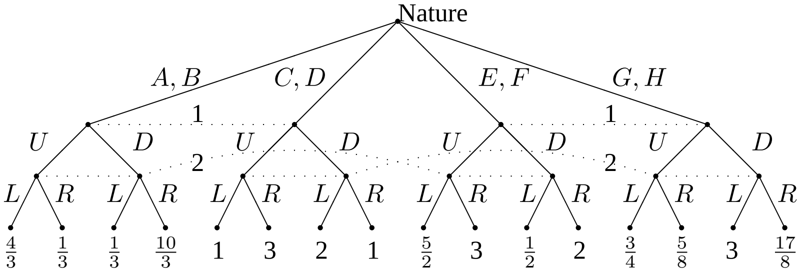
Definition 1: Example



Definition 2: Extensive Form with Chance Moves

- Add an agent, “Nature,” who follows a commonly known mixed strategy.
- Thus, reduce Bayesian games to extensive form games of imperfect information.
- This definition is cumbersome for the same reason that IIEF is a cumbersome way of representing matrix games like Prisoner’s dilemma
  - however, it makes sense when the agents really do move sequentially, and at least occasionally observe each other’s actions.

Definition 2: Example



Definition 3: Epistemic Types

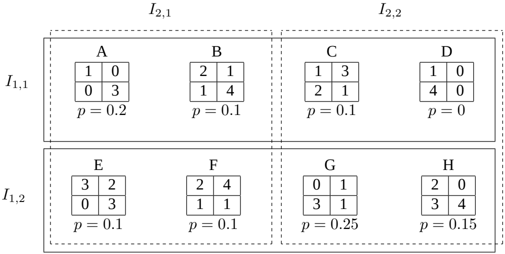
- Directly represent uncertainty over utility function using the notion of epistemic type.

Definition

A Bayesian game is a tuple (N, A, Θ, p, u) where

- N is a set of agents,
- $A = (A_1, \ldots, A_n)$, where A_i is the set of actions available to player i,
- $\Theta = (\Theta_1, \ldots, \Theta_n)$, where Θ_i is the type space of player i,
- $p : \Theta \rightarrow [0, 1]$ is the common prior over types,
- $u = (u_1, \ldots, u_n)$, where $u_i : A \times \Theta \rightarrow \mathbb{R}$ is the utility function for player i.

Definition 3: Example



a_1	a_2	θ_1	θ_2	u_1
U	L	$\theta_{1,1}$	$\theta_{2,1}$	4/3
U	L	$\theta_{1,1}$	$\theta_{2,2}$	1
U	L	$\theta_{1,2}$	$\theta_{2,1}$	5/2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	3/4
U	R	$\theta_{1,1}$	$\theta_{2,1}$	1/3
U	R	$\theta_{1,1}$	$\theta_{2,2}$	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	3
U	R	$\theta_{1,2}$	$\theta_{2,2}$	5/8

a_1	a_2	θ_1	θ_2	u_1
D	L	$\theta_{1,1}$	$\theta_{2,1}$	1/3
D	L	$\theta_{1,1}$	$\theta_{2,2}$	2
D	L	$\theta_{1,2}$	$\theta_{2,1}$	1/2
D	L	$\theta_{1,2}$	$\theta_{2,2}$	3
D	R	$\theta_{1,1}$	$\theta_{2,1}$	10/3
D	R	$\theta_{1,1}$	$\theta_{2,2}$	1
D	R	$\theta_{1,2}$	$\theta_{2,1}$	2
D	R	$\theta_{1,2}$	$\theta_{2,2}$	17/8