The Folk Theorem

CPSC 532A Lecture 12

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Lecture Overview

Perfect Recall: mixed and behavioral strategies coincide

Intuitively, no player forgets any information he knew about moves made so far

Definition

Player i has perfect recall in an imperfect-information game G if for any two nodes h, h' that are in the same information set for player i, for any path $h_0, a_0, h_1, a_1, h_2, \ldots, h_n, a_n, h$ from the root of the game to h (where the h_j are decision nodes and the a_j are actions) and any path $h_0, a'_0, h'_1, a'_1, h'_2, \ldots, h'_m, a'_m, h'$ from the root to h' it must be the case that:

1. $n = m$
2. For all $0 \leq j \leq n$, h_j and h'_j are in the same equivalence class for player i.
3. For all $0 \leq j \leq n$, if $\rho(h_j) = i$ (that is, h_j is a decision node of player i), then $a_j = a'_j$.

G is a game of perfect recall if every player has perfect recall in it.

Perfect Recall

Clearly, every perfect-information game is a game of perfect recall.

Theorem (Kuhn, 1953)

In a game of perfect recall, any mixed strategy of a given agent can be replaced by an equivalent behavioral strategy, and any behavioral strategy can be replaced by an equivalent mixed strategy. Here two strategies are equivalent in the sense that they induce the same probabilities on outcomes, for any fixed strategy profile (mixed or behavioral) of the remaining agents.

- Corollary: in games of perfect recall the set of Nash equilibria does not change if we restrict ourselves to behavioral strategies.

Finitely Repeated Games

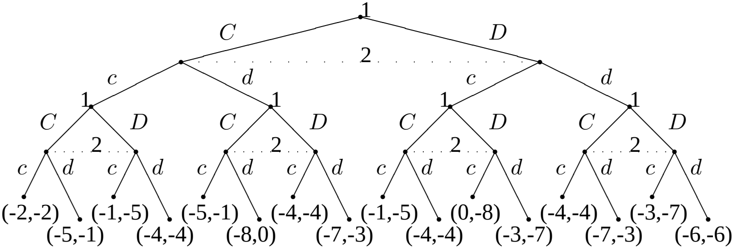
- Everything is straightforward if we repeat a game a finite number of times
- we can write the whole thing as an extensive-form game with imperfect information
  - at each round players don't know what the others have done; afterwards they do
  - overall payoff function is additive: sum of payoffs in stage games

Example

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

$\Rightarrow$

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$



Notes

- Observe that the strategy space is much richer than it was in the NF setting
- Repeating a Nash strategy in each stage game will be an equilibrium in behavioral strategies (called a stationary strategy)
- In general strategies adopted can depend on actions played so far
- We can apply backward induction in these games when the normal form game has a dominant strategy.

Infinitely Repeated Games

Definition

Given an infinite sequence of payoffs $r_1, r_2, \ldots$ for player i, the average reward of i is $\lim_{k\to\infty} \Sigma_{j=1}^{k} r_j/k$.

Definition

Given an infinite sequence of payoffs $r_1, r_2, \ldots$ for player i and a discount factor β with $0 \leq \beta \leq 1$, the future discounted rewards of i is $\sum_{j=1}^{\infty} \beta^j r_j$.

Nash Equilibria

- With an infinite number of equilibria, what can we say about Nash equilibria?
  - we won't be able to construct an induced normal form and then appeal to Nash's theorem to say that an equilibrium exists
  - Nash's theorem only applies to finite games
- Furthermore, with an infinite number of strategies, there could be an infinite number of pure-strategy equilibria!
- It turns out we can characterize a set of payoffs that are achievable under equilibrium, without having to enumerate the equilibria.

Lecture Overview

Recap

Folk Theorem

Definitions

- Consider any n-player game $G = (N, (A_i), (u_i))$ and any payoff vector $r = (r_1, r_2, \ldots, r_n)$.
- Let $v_i = \min\limits_{s_{-i} \in S_{-i}} \max\limits_{s_i \in S_i} u_i(s_{-i}, s_i)$.
  - the amount of utility i can get when $-i$ play a minmax strategy against him

Definition

A payoff profile r is enforceable if $r_i \geq v_i$.

Definition

A payoff profile r is feasible if there exist rational, non-negative values α_a such that for all i, we can express r_i as $\sum_{a \in A} \alpha u_i(a)$, with $\sum_{a \in A} \alpha_a = 1$.

- a payoff profile is feasible if it is a convex, rational combination of the outcomes in G.

Folk Theorem

Theorem (Folk Theorem)

Consider any n-player game G and any payoff vector $(r_1, r_2, \ldots, r_n)$.

1. *If r is the payoff in any Nash equilibrium of the infinitely repeated G with average rewards, then for each player i, r_i is enforceable.*
2. *If r is both feasible and enforceable, then r is the payoff in some Nash equilibrium of the infinitely repeated G with average rewards.*

Folk Theorem (Part 1)

Payoff in Nash → enforceable

Part 1: Suppose r is not enforceable, i.e. $r_i < v_i$ for some i. Then consider a deviation of this player i to $b_i(s_{-i}(h))$ for any history h of the repeated game, where b_i is any best-response action in the stage game and $s_{-i}(h)$ is the equilibrium strategy of other players given the current history h. By definition of a minmax strategy, player i will receive a payoff of at least v_i in every stage game if he adopts this strategy, and so i's average reward is also at least v_i. Thus i cannot receive the payoff $r_i < v_i$ in any Nash equilibrium.

Folk Theorem (Part 2)

Feasible and enforceable $\rightarrow$ Nash

Part 2: Since r is a feasible enforceable payoff profile, we can write it as $r_i = \sum_{a \in A} (\frac{\beta_a}{\gamma}) u_i(a)$, where β_a and γ are non-negative integers. (Recall that α_a were required to be rational. So we can take γ to be their common denominator.) Since the combination was convex, we have $\gamma = \sum_{a \in A} \beta_a$.

We're going to construct a strategy profile that will cycle through all outcomes $a \in A$ of G with cycles of length γ, each cycle repeating action a exactly β_a times. Let (a^t) be such a sequence of outcomes. Let's define a strategy s_i of player i to be a trigger version of playing (a^t): if nobody deviates, then s_i plays a_i^t in period t. However, if there was a period t' in which some player $j \neq i$ deviated, then s_i will play $(p_{-j})_i$, where (p_{-j}) is a solution to the minimization problem in the definition of v_j.

Folk Theorem (Part 2)

Feasible and enforceable $\rightarrow$ Nash

First observe that if everybody plays according to s_i, then, by construction, player i receives average payoff of r_i (look at averages over periods of length γ). Second, this strategy profile is a Nash equilibrium. Suppose everybody plays according to s_i, and player j deviates at some point. Then, forever after, player j will receive his $\min\max$ payoff $v_j \leq r_j$, rendering the deviation unprofitable.