Perfect Recall and Repeated Games

CPSC 532A Lecture 11

October 17, 2006

Lecture Overview

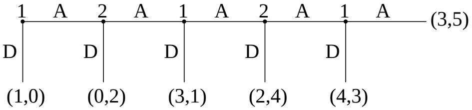
Recap

Perfect Recall

Repeated Games

Infinitely Repeated Games

Backward Induction



- What happens when we use this procedure on Centipede?
  - In the only equilibrium, player 1 goes down in the first move.
  - However, this outcome is Pareto-dominated by all but one other outcome.
- Two considerations:
  - practical: human subjects don't go down right away
  - theoretical: what should you do as player 2 if player 1 doesn't go down?
    - SPE analysis says to go down. However, that same analysis says that P1 would already have gone down. How do you update your beliefs upon observation of a measure zero event?
    - but if player 1 knows that you'll do something else, it is rational for him not to go down anymore... a paradox
    - there's a whole literature on this question

Imperfect Information Extensive Form

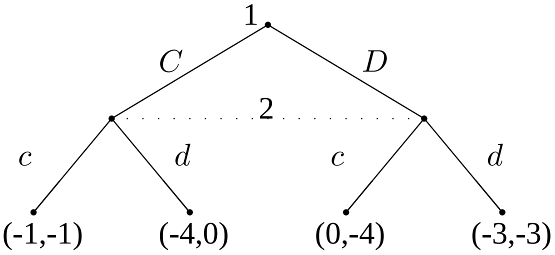
Definition

An imperfect-information games (in extensive form) is a tuple $(N, A, H, Z, \chi, \rho, \sigma, u, I)$, where

- $(N, A, H, Z, \chi, \rho, \sigma, u)$ is a perfect-information extensive-form game, and
- $I = (I_1, \ldots, I_n)$, where $I_i = (I_{i,1}, \ldots, I_{i,k_i})$ is an equivalence relation on (that is, a partition of) $\{h \in H : \rho(h) = i\}$ with the property that $\chi(h) = \chi(h')$ and $\rho(h) = \rho(h')$ whenever there exists a j for which $h \in Ii, j$ and $h' \in Ii, j$.

Normal-form games

- We can represent any normal form game.



  - Note that it would also be the same if we put player 2 at the root node.

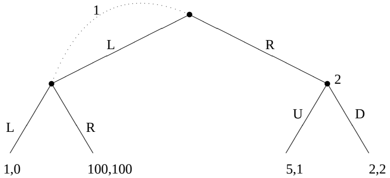
Induced Normal Form

- Same as before: enumerate pure strategies for all agents
- Mixed strategies are just mixtures over the pure strategies as before.
- Nash equilibria are also preserved.
- Note that we've now defined both mapping from NF games to IIEF and a mapping from IIEF to NF.
  - what happens if we apply each mapping in turn?
  - we might not end up with the same game, but we do get one with the same strategy spaces and equilibria.

Randomized Strategies

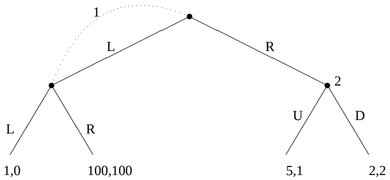
- It turns out there are two meaningfully different kinds of randomized strategies in imperfect information extensive form games
  - mixed strategies
  - behavioral strategies
- Mixed strategy: randomize over pure strategies
- Behavioral strategy: independent coin toss every time an information set is encountered

Games of imperfect recall



- What is the space of pure strategies in this game?
  - 1: (L, R); 2: (U, D)
- What is the mixed strategy equilibrium?
  - Observe that D is dominant for 2. R, D is better for 1 than L, D, so R, D is an equilibrium.

Games of imperfect recall



- What is an equilibrium in behavioral strategies?
  - again, D strongly dominant for 2
  - if 1 uses the behavioural strategy $(p, 1-p)$, his expected utility is $1 * p2 + 100 * p(1-p) + 2 * (1-p)$
  - simplifies to $-99p^2 + 98p + 2$
  - maximum at $p = 98/198$
  - thus equilibrium is $(98/198, 100/198), (0, 1)$
- Thus, we can have behavioral strategies that are different from mixed strategies.

Lecture Overview

Recap

Perfect Recall

Repeated Games

Infinitely Repeated Games

Perfect Recall: mixed and behavioral strategies coincide

Intuitively, no player forgets any information he knew about moves made so far

Definition

Player i has perfect recall in an imperfect-information game G if for any two nodes h, h' that are in the same information set for player i, for any path $h_0, a_0, h_1, a_1, h_2, \ldots, h_n, a_n, h$ from the root of the game to h (where the h_j are decision nodes and the a_j are actions) and any path $h_0, a'_0, h'_1, a'_1, h'_2, \ldots, h'_m, a'_m, h'$ from the root to h' it must be the case that:

1. $n = m$
2. For all $0 \leq j \leq n$, h_j and h'_j are in the same equivalence class for player i.
3. For all $0 \leq j \leq n$, if $\rho(h_j) = i$ (that is, h_j is a decision node of player i), then $a_j = a'_j$.

G is a game of perfect recall if every player has perfect recall in it.

Perfect Recall

Clearly, every perfect-information game is a game of perfect recall.

Theorem (Kuhn, 1953)

In a game of perfect recall, any mixed strategy of a given agent can be replaced by an equivalent behavioral strategy, and any behavioral strategy can be replaced by an equivalent mixed strategy. Here two strategies are equivalent in the sense that they induce the same probabilities on outcomes, for any fixed strategy profile (mixed or behavioral) of the remaining agents.

- Corollary: in games of perfect recall the set of Nash equilibria does not change if we restrict ourselves to behavioral strategies.

Lecture Overview

Recap

Perfect Recall

Repeated Games

Infinitely Repeated Games

Introduction

- Play the same normal-form game over and over
  - each round is called a "stage game"
- Questions we'll need to answer:
  - what will agents be able to observe about others' play?
  - how much will agents be able to remember about what has happened?
  - what is an agent's utility for the whole game?
- Some of these questions will have different answers for finitely- and infinitely-repeated games.

Finitely Repeated Games

- Everything is straightforward if we repeat a game a finite number of times
- we can write the whole thing as an extensive-form game with imperfect information
  - at each round players don't know what the others have done; afterwards they do
  - overall payoff function is additive: sum of payoffs in stage games

Notes

- Observe that the strategy space is much richer than it was in the NF setting
- Repeating a Nash strategy in each stage game will be an equilibrium in behavioral strategies (called a stationary strategy)
- In general strategies adopted can depend on actions played so far
- We can apply backward induction in these games when the normal form game has a dominant strategy.

Example

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

$\Rightarrow$

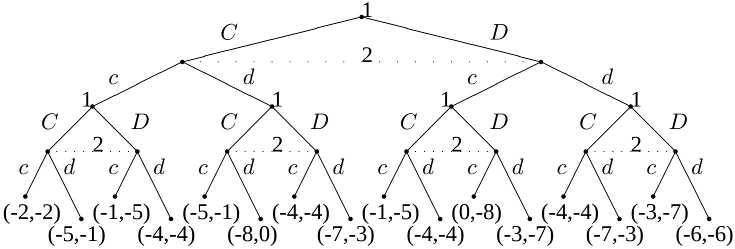
	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

Example

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$

$\Rightarrow$

	C	D
C	$-1, -1$	$-4, 0$
D	$0, -4$	$-3, -3$



Lecture Overview

Recap

Perfect Recall

Repeated Games

Infinitely Repeated Games

Infinitely Repeated Games

- Consider an infinitely repeated game in extensive form:
  - an infinite tree!
- Thus, payoffs cannot be attached to terminal nodes, nor can they be defined as the sum of the payoffs in the stage games (which in general will be infinite).

Definition

Given an infinite sequence of payoffs $r_1, r_2, \ldots$ for player i, the average reward of i is $\lim_{k\to\infty} \Sigma_{j=1}^{k} r_j/k$.

Discounted reward

Definition

Given an infinite sequence of payoffs $r_1, r_2, \ldots$ for player i and a discount factor β with $0 \leq \beta \leq 1$, the future discounted rewards of i is $\sum_{j=1}^{\infty} \beta^j r_j$.

- Interpreting the discount factor:
  1. the agent cares more about his well-being in the near term than in the long term
  2. the agent cares about the future just as much as the present, but with probability $1 - \beta$ the game will end in any given round.
- The analysis of the game is the same under both perspectives.

Strategy Space

- What is a pure-strategy in an infinitely-repeated game?

Strategy Space

- What is a pure-strategy in an infinitely-repeated game?
  - a choice of action at every decision point
  - here, that means an action at every stage game
  - ...which is an infinite number of actions!
- Some famous strategies (repeated PD):
  - Tit-for-tat: Start out cooperating. If the opponent defected, defect in the next round. Then go back to cooperation.
  - Trigger: Start out cooperating. If the opponent ever defects, defect forever.

Nash Equilibria

- With an infinite number of equilibria, what can we say about Nash equilibria?
  - we won't be able to construct an induced normal form and then appeal to Nash's theorem to say that an equilibrium exists
  - Nash's theorem only applies to finite games
- Furthermore, with an infinite number of strategies, there could be an infinite number of pure-strategy equilibria!
- It turns out we can characterize a set of payoffs that are achievable under equilibrium, without having to enumerate the equilibria.