CPSC 532A, Fall 2006
Homework #3

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1. **[30 points]** Consider a potentially infinite outcome space $\mathcal{O} \subset [0, 1]$, and a finite set N of n agents. Denote the utility of an agent with type θ_i for outcome o as $u_i(o, \theta_i)$. Constrain the utility functions so that every agent has some unique, most-preferred outcome $b(\theta_i) \in \mathcal{O}$, and so that $|o' - b(\theta_i)| < |o'' - b(\theta_i)|$ implies that $u_i(o', \theta_i) > u_i(o'', \theta_i)$.

   Consider a direct mechanism which asks every agent to declare his most-preferred outcome and then selects the median outcome. (If there are an even number of agents, the mechanism chooses the larger of the two middle outcomes.)

   (a) Prove that truthtelling is a dominant strategy.

   (b) Prove that the mechanism selects a Pareto optimal outcome.

   (c) Prove that if the mechanism designer submits $n - 1$ "dummy preferences" of her own with any values she likes, and then runs the same mechanism on the $2n-1$ preferences, the dominant strategy is preserved.

   (d) As described so far, the mechanism selects the $\lceil \frac{n}{2} \rceil^{\text{th}}$-order statistic of the declared preferences. (The k^{th}-order statistic of a set of numbers is the k^{th}-largest number in the set.) Explain how to select dummy preferences in such a way that the mechanism selects the k^{th}-order statistic of the agents' declared preferences for any $k \in \{1, \ldots, n\}$. Of course, the dummy preferences must be set in a way that does not depend on the specific declarations made by the agents.

2. **[30 points]** Suppose you have some object that each of n agents desires, but which you do not value. Assume that each agent i values it at v_i, with v_i's drawn independently and uniformly from some positive real line interval, say $[0, 10^{100}]$. Although you do not desire the object and also do not care about the actual values of the v_i's, you need to compute $\sqrt{v_i}$ for each i.

   Unfortunately, you face two problems. First, agents are not inclined to just reveal to you anything about their v_i's. Second, your computer is costly to operate. It costs you 1 unit to determine the greater of two values, 2 units to perform any basic arithmetic operation $(+, -, \times, /)$, and anything more complicated (such as $\sqrt{x}$) costs 20. The (accurate) current time of day can be observed without cost.

   (a) How much would it cost to compute $\sqrt{v_i}$ for each i using a straightforward VCG mechanism? (When computing cost, ignore the revenue that the auction will generate.) Hint: this part is very easy.

   (b) Your answer above gives an upper bound on the cost of computing the square roots of the valuations. Design an incentive compatible, dominant strategy ("strategy-proof") direct mechanism that will allow you to compute all $\sqrt{v_i}$ at *minimal* cost. Assume that agents can do computations for free. Make sure you specify all the components of the mechanism: players, actions, outcomes, mappings from actions to outcomes. Explain why your mechanism is strategy-proof. Specify the algorithms that you will use to implement those mappings. Give your mechanism's total computation cost (or a reasonable upper bound on it). You do not need to prove that your mechanism achieves minimal cost.

(Hint: think about the revelation principle.)

(c) In the previous part you were restricted to direct mechanisms. Show that an *indirect* mechanism can achieve even lower cost.

3. **[30 points]** The VCG mechanism does not violate the Myerson-Satterthwaite Theorem because it is not budget balanced for general quasilinear preferences. But this seems like an easy enough problem to solve—we can just evenly redistribute any money that was collected by the mechanism (or, tax all agents equally if the net payment to the agents was positive). Below is a proposed, budget-balanced version of the VCG Mechanism. It converts what was p_i into a temporary variable t_i. Then, the new payments p_i contains an equal redistribution of the sum of the original payments.

   The *Budget-Balanced VCG Mechanism* is a direct mechanism $M(\hat{v}) = (x(\hat{v}), p_1(\hat{v}), \ldots, p_n(\hat{v}))$, where

   - $x(\hat{v}) = \arg\max_{x \in X} \sum_{i \in N} \hat{v}_i(x)$, and
   - $t_i(\hat{v}) = \max_{o \in O_{-i}} \sum_{j \neq i} \hat{v}_j(o) - \sum_{j \neq i} \hat{v}_j(x)$
   - $p_i(\hat{v}) = t_i - \frac{1}{n} \sum_i t_i(\hat{v})$

   (a) Show that this mechanism is not incentive compatible. One option is to prove this directly. Alternately, you can give two valuation functions for an agent i (one that gives his true valuation (v_i), and alternative one (v_i')) and a set of declared valuation functions ($\hat{v}_j, \hat{v}_k, \ldots$) for as many agents other than i that you need, and show that if all other agents declare these valuations, the utility for agent i is higher if he declares v_i' instead of v_i.

   (b) Although our first mechanism failed, we can use a similar idea to make VCG budget-balanced *ex-ante*. Assume that bidders valuations v_i are randomly drawn from some joint commonly known distribution. Consider the following mechanism:

   Ex-Ante Budget-Balanced VCG Mechanism is a direct mechanism $M(\hat{v}) = (x(\hat{v}), p_1(\hat{v}), \ldots, p_n(\hat{v}))$, where

   - $x(\hat{v}) = \arg\max_{x \in X} \sum_{i \in N} \hat{v}_i(x)$, and
   - $t_i(\hat{v}) = \max_{o \in O_{-i}} \sum_{j \neq i} \hat{v}_j(o) - \sum_{j \neq i} \hat{v}_j(x(\hat{v}))$
   - $p_i(\hat{v}) = t_i(\hat{v}) - \frac{1}{n} \sum_j E_v[t_j(v)]$

   Prove that truth-telling is a dominant strategy in this new mechanism.

   (c) Show that this mechanism is *ex-ante* budged-balanced on expectation (i.e. expected total payment by all agents is zero).

4. **[20 points]** Consider a first-price auction with two bidders. Assume that they have IPV valuations drawn uniformly from the interval $[0, 10]$, and that they are risk-neutral. In class (and in the textbook) we saw that $s_1(v_1) = \frac{1}{2}v_1$ and $s_2(v_2) = \frac{1}{2}v_2$ together form a Bayes-Nash equilibrium for this game.

   (a) Assuming that bidder 2 is instead using the bidding strategy $s_2(v_2) = v_2$ (i.e., she bids his valuation), what is the best response bidding strategy $s_1(v_1)$ for player 1? Show your work at a level similar to that used in class and in the textbook.

(b) Now consider instead a *second-price* auction. However, suppose the mechanism has a buggy implementation of max: most of the time the mechanism works correctly, but with some probability p which is strictly less than $1/3$ it awards the object to the second-highest bidder (instead of the highest bidder). In all cases it correctly calculates price as the second highest bid. Assuming that bidder 2 is still bidding truthfully, compute the best-response strategy for bidder 1. Is it still truthful? Show your work at the same level as in your answer to the first part.

5. **[26 points]** Consider the following lobbying problem. There are n different companies, each of which wants the government to pass legislation that will benefit that company and will have no direct effect on the other companies. If the legislation that favours company i is passed, i's profit will be v_i; otherwise it will be 0. In order to try to influence government policy, each company i considers making a donation of some amount d_i to the government. Let's consider the case where all v_i are independent random variables distributed uniformly on $[0, 1]$. Somewhat cynically, the government will pass the single piece of legislation that benefits its biggest donor; of course, it will keep all the donations it receives.

   (a) Model this problem as an auction. State all the relevant assumptions that you make in building this model, and explain why they are reasonable. You do not have to restate the assumptions given above. (This part is not hard, and requires only a short answer.)

   (b) Find a symmetric Bayes-Nash equilibrium of this game. You may assume that for this game there exists a symmetric, pure-strategy equilibrium for which the bid amount is a monotonically increasing function of the agent's valuation.

Academic Honesty Form

For this assignment, it is acceptable to collaborate with other students provided that you write up your solutions independently. The only reference materials that you can use are the course notes and textbook, and the reference textbooks listed on the course web page. In particular, getting help from students or course materials from previous years is not acceptable.

List any people you collaborated with:

List any non-course materials you refered to:

Signature:

Fill in this page and include it with your assignment submission.