Decision Theory: Markov Decision Processes

CPSC 322 – Decision Theory 3

Textbook §12.5
Policies

- A policy specifies what an agent should do under each circumstance.
- A policy is a sequence $\delta_1, \ldots, \delta_n$ of decision functions

$$\delta_i : \text{dom}(pD_i) \rightarrow \text{dom}(D_i).$$

This policy means that when the agent has observed $O \in \text{dom}(pD_i)$, it will do $\delta_i(O)$. 
Possible world \( \omega \) satisfies policy \( \delta \), written \( \omega \models \delta \), if the world assigns the value to each decision node that the policy specifies.

The expected utility of policy \( \delta \) is

\[
\mathbb{E}(U|\delta) = \sum_{\omega \models \delta} P(\omega)U(\omega)
\]

An optimal policy is one with the highest expected utility:

\[
\delta^* \in \arg \max_{\delta} \mathbb{E}(U|\delta).
\]
Counting Policies

- If a decision $D$ has $k$ binary parents, how many assignments of values to the parents are there? $2^k$
- If there are $b$ possible actions, how many different decision functions are there? $b^{2^k}$
- If there are $d$ decisions, each with $k$ binary parents and $b$ possible actions, how many policies are there? $(b^{2^k})^d$
Decision Network for the Alarm Problem

- Tampering
- Fire
- Alarm
- Smoke
- Leaving
- Check Smoke
- Report
- See Smoke
- Call
- Utility

Decision Theory: Markov Decision Processes

CPSC 322 – Decision Theory 3, Slide 6
Lecture Overview

1. Recap
2. Finding Optimal Policies
3. Value of Information, Control
4. Markov Decision Processes
5. Rewards and Policies
Finding the optimal policy

- **Remove** all variables that are not ancestors of a value node.
- Create a factor for each conditional probability table and a factor for the utility.
- **Sum out** variables that are not parents of a decision node.
- Select a variable $D$ that is only in a factor $f$ with (some of) its parents.
  - this variable will be one of the decisions that is made **latest**
- Eliminate $D$ by **maximizing**. This returns:
  - the optimal decision function for $D$, $\arg \max_D f$
  - a new factor to use in VE, $\max_D f$
- Repeat till there are no more decision nodes.
- **Sum out** the remaining random variables. Multiply the factors: this is the expected utility of the optimal policy.
Recall: If there are $d$ decisions, each with $k$ binary parents and $b$ possible actions, there are $\left(b^{2k}\right)^d$ policies.

Doing variable elimination lets us find the optimal policy after considering only $d \cdot b^{2k}$ policies.

The dynamic programming algorithm is much more efficient than searching through policy space.

However, this complexity is still doubly-exponential—we’ll only be able to handle relatively small problems.
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Value of Information

- How much you should be prepared to pay for a sensor?
- E.g., how much is a better weather forecast worth?

**Definition (Value of Information)**

The value of information $X$ for decision $D$ is the utility of the the network with an arc from $X$ to $D$ minus the utility of the network without the arc.

- The value of information is always non-negative.
- It is positive only if the agent changes its action depending on $X$. 
We could ask about the value of information for Smoke
Value of Control

- How useful is it to be able to set a random variable?

**Definition (Value of Control)**

The value of control of a variable $X$ is the value of the network when you make $X$ a decision variable minus the value of the network when $X$ is a random variable.

- You need to be explicit about what information is available when you control $X$.
  - If you control $X$ without observing, controlling $X$ can be worse than observing $X$.
  - If you keep the parents the same, the value of control is always non-negative.
We could ask about the value of control for Tampering.
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Agents as Processes

Agents carry out actions:

- forever: infinite horizon
- until some stopping criteria is met: indefinite horizon
- finite and fixed number of steps: finite horizon
Decision-theoretic Planning

What should an agent do under these different planning horizons, when

- actions can be noisy
  - the outcome of an action can't be fully predicted
  - there is a stationary, Markovian model that specifies the (probabilistic) outcome of actions
- the world (i.e., state) is fully observable
- the agent periodically gets rewards (and punishments) and wants to maximize its rewards received
Stationary Markov chain

Start with a stationary Markov chain.

- Recall: a stationary Markov chain is when for all $t > 0$, $P(S_{t+1}|S_t) = P(S_{t+1}|S_0, \ldots, S_t)$.
- We specify $P(S_0)$ and $P(S_{t+1}|S_t)$. 

A Markov decision process augments a stationary Markov chain with actions and values:
Markov Decision Processes

Definition (Markov Decision Process)

A Markov Decision Process (MDP) is a 5-tuple \( \langle S, A, P, R, s_0 \rangle \), where each element is defined as follows:

- \( S \): a set of states.
- \( A \): a set of actions.
- \( P(S_{t+1}|S_t, A_t) \): the dynamics.
- \( R(S_t, A_t, S_{t+1}) \): the reward. The agent gets a reward at each time step (rather than just a final reward).
  - \( R(s, a, s') \) is the reward received when the agent is in state \( s \), does action \( a \) and ends up in state \( s' \).
- \( s_0 \): the initial state.
Example: Simple Grid World

- Actions: up, down, left, right.
- 100 states corresponding to the positions of the robot.
- Robot goes in the commanded direction with probability 0.7, and one of the other directions with probability 0.1.
- If it crashes into an outside wall, it remains in its current position and has a reward of $-1$.
- Four special rewarding states; the agent gets the reward when leaving.
The planning horizon is how far ahead the planner can need to look to make a decision.

- The robot gets flung to one of the corners at random after leaving a positive (+10 or +3) reward state.
  - the process never halts
  - infinite horizon

- The robot gets +10 or +3 entering the state, then it stays there getting no reward. These are absorbing states.
  - The robot will eventually reach the absorbing state.
  - indefinite horizon
What information is available when the agent decides what to do?

- **fully-observable MDP** the agent gets to observe $S_t$ when deciding on action $A_t$.

- **partially-observable MDP (POMDP)** the agent has some noisy sensor of the state. It needs to remember its sensing and acting history.

We’ll only consider (fully-observable) MDPs.
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Suppose the agent receives the sequence of rewards $r_1, r_2, r_3, r_4, \ldots$. What value should be assigned?

- **total reward:**
  
  \[
  V = \sum_{i=1}^{\infty} r_i
  \]

- **average reward:**
  
  \[
  V = \lim_{n \to \infty} \frac{r_1 + \cdots + r_n}{n}
  \]

- **discounted reward:**
  
  \[
  V = \sum_{i=1}^{\infty} \gamma^{i-1} r_i
  \]

- \(\gamma\) is the discount factor, \(0 \leq \gamma \leq 1\)